

COLORING PLANAR GRAPHS VIA COLORED PATHS IN THE ASSOCIAHEDRA

GARRY BOWLIN AND MATTHEW G. BRIN

Abstract. Hassler Whitney’s theorem of 1931 reduces the task of finding proper, vertex 4-colorings of triangulations of the 2-sphere to finding such colorings for the class \mathfrak{H} of triangulations of the 2-sphere that have a Hamiltonian circuit. This has been used by Whitney and others from 1936 to the present to find equivalent reformulations of the 4 Color Theorem (4CT). Recently there has been activity to try to use some of these reformulations to find a shorter proof of the 4CT. Every triangulation in \mathfrak{H} has a dual graph that is a union of two binary trees with the same number of leaves. Elements of a group known as Thompson’s group F are equivalence classes of pairs of binary trees with the same number of leaves. This paper explores this resemblance and finds that some recent reformulations of the 4CT are essentially attempting to color elements of \mathfrak{H} using expressions of elements of F as words in a certain generating set for F . From this, we derive information about not just the colorability of certain elements of \mathfrak{H} , but also about all possible ways to color these elements. Because of this we raise (and answer some) questions about enumeration, and we conjecture that certain elements of an extension E of the group F “parametrize” the set of all colorings of the elements of \mathfrak{H} that use all four colors.

1. Introduction	2
2. Starting from Whitney’s theorem	9
3. Trees	12
4. The associahedra	16
5. Color	20
6. Colored rotations and colored paths	24
7. Groups	27
8. Colors and the group operations	34
9. Rigid colorings	38
10. Color graphs, zero sets, shadow patterns, and long paths	40
11. Sign structures	43
12. Acceptable color vectors	51
13. Patterns	52
14. Higher genera and Thompson’s group V	55
15. Enumeration	58
16. The end	67
References	68
Index	70

Date: January 12, 2013.

AMS Classification (2010): primary 05C15, secondary 20F05, 05C10, 05C30, 05C20.

1. INTRODUCTION

This paper combines algebraic structures (that of a specific finitely presented group) with the problem of coloring planar maps. This paper can be viewed either as a search for a shorter proof of the Four Color Theorem or as an exploration of the colorings that must exist given the truth of the theorem. We present a mechanism that finds all colorings. We describe the mechanism, present some of its properties, and discuss questions of enumeration that it leads to.

In this introduction, Section 1.1 gives enough background to tie the items in the title of the paper together, Section 1.2 describes some of our results, and Section 1.3 lists the topics in the order that they are developed in the paper.

1.1. Background. A map M is an embedding of a finite graph (whose image we will call the underlying graph of the map and which we will still denote by M) into the 2-sphere S^2 . A proper, face 4-coloring of M is a function from the faces of M to a set of four colors so that any two different faces that share an edge are assigned different colors. Finding a proper, face 4-coloring for M is equivalent to finding a proper, vertex 4-coloring (no two adjacent vertices receive the same color) of the graph dual to M in S^2 . The 4 Color Theorem (4CT) says that such a 4-coloring can always be found.

The 4CT has been proven in [1], [2] and [18] and the proof has been verified by [10]. Since these proofs and verifications use large amounts of computer calculations, shorter proofs have been sought.

It is standard to start a discussion of the 4CT by reducing the set of graphs to be considered. One quick reduction is to the class of cubic graphs. A further reduction using the shrinking of edges eliminates loops in the dual graph in S^2 . See Section 3-2 of [19]. Shrinking edges also can eliminate parallel edges in the dual graph, but we want to do this elimination using a different technique since the different technique is more relevant to us.

1.1.1. Two reductions. For the rest of this introduction all graphs have no loops.

Let us assume that T is a finite graph on S^2 all of whose faces are triangles, but that fails to be a triangulation because it has parallel edges. Let C be a circuit consisting of two parallel edges. Each complementary domain of C in S^2 contains a vertex of T since every face of T is a triangle. There are two graphs we can get by deleting all vertices and edges of T that are in one complementary domain of C . Call these graphs the result of “cutting T along C .” Each of these can be properly vertex 4-colored by an inductive hypothesis. In each of these graphs, the two colors given to the vertices of C are different. Thus there are exactly two permutations of the colors of one of the graphs that allow us to put the two colorings together to color all of T .

The phrase “exactly two permutations” is relevant in the following way. We can clearly reduce T by a finite number of such cuts to a finite set of “prime” pieces that are true triangulations in that they have no parallel edges by identifying the two edges of a cut each time we make a cut. The original T can be recovered from such pieces if it is remembered where the cuts were and how the pieces fit together. It is also clear that if there are k pieces and the various pieces have, respectively, n_1, n_2, \dots, n_k different proper, vertex 4-colorings up to permutations of the colors,

then T has

$$(1) \quad 2^{k-1} \prod_{i=1}^k n_i$$

different proper, vertex 4-colorings up to permutations of the colors.

Just as a circuit of length 2 must use two different colors for its vertices, a circuit of length 3 must use three different colors for its vertices. We base a second reduction on this.

If we ignore parallel edges and turn our attention to separating circuits of length 3, then a similar analysis takes place and we reduce T to a finite number “irreducible” pieces that have no separating circuits of length 3. No identification of edges is needed since the edges in the cut will form a triangular face in each of the two subgraphs created by a cut. If there are j such pieces and they have (up to permutations of the colors), respectively, m_1, m_2, \dots, m_j different proper, vertex 4-colorings, then T has $m_1 m_2 \cdots m_j$ different proper vertex 4-colorings up to permutations of the colors.

It follows that if we completely understand proper, vertex 4-colorings of graphs in S^2 whose faces are triangles and which are both irreducible and prime, then we understand all proper, vertex 4-colorings of all graphs in S^2 whose faces are triangles.

1.1.2. Whitney’s theorem. In the previous setting, a piece that is both “prime” and “irreducible” is a finite graph in S^2 with triangular faces, with no loops, no parallel edges and no separating circuits of length 3. Let P be such a piece. By the main theorem of [25], P must have a Hamiltonian circuit C . A restatement of Whitney’s theorem is given as Theorem 4-7 in [19]. If P has n vertices, then cutting P along C will give two triangulated polygons of n sides and edges.

We take this conclusion to be the starting point of the paper. We consider maps M in S^2 whose underlying graph is cubic (so that the dual has only triangular faces) and whose dual has a Hamiltonian circuit and no loops. Since we base our definition on the conclusion of Whitney’s theorem, we lose the parts of the hypotheses that are not mentioned. In particular, the dual may have parallel edges and separating circuits of length 3. It turns out that the parallel edges will concern us in what follows, but we will not worry at all about separating circuits of length 3. Perhaps further studies taking these circuits into account could be fruitful.

1.1.3. The setting and the dual views. In the following, “triangulated polygon” will always assume that all the vertices are on the boundary of the polygon. That is, no extra vertices were introduced by the triangulation.

We study those maps M in S^2 that are in a certain class \mathfrak{W} . For M to be in \mathfrak{W} , its dual graph in S^2 must have no loops, must have triangles for faces and must have a Hamiltonian circuit. The graph M is thus cubic and connected.

The Hamiltonian circuit in the dual of M will be called the equator. It cuts through each face of M in a single arc and we require that it passes through no vertex of M and be transverse to the edges of M . The intersection D of M with a complementary domain of the equator will be a tree since if D is not connected, then some face of M will hit the equator twice and if D is not simply connected, then some face of M will not hit the equator. The resulting tree is binary since all

vertices other than the leaves have degree three. Thus M consists of two binary trees that meet each other in their leaves along the equator.

The dual graph T of M will fall into two triangulated polygons C_1 and C_2 bounded by the equator. While T may have parallel edges, no two parallel edges will reside in one of the C_i since there would then have to be a face that is not a triangle. Thus two parallel edges must have one edge in C_1 and the other in C_2 . The restriction of T to each polygon will then be a true triangulation of that polygon.

We make the setting more specific. If we make S^2 the set of points in \mathbf{R}^3 of distance one from the origin, we can insist that the Hamiltonian circuit lie on the circle in S^2 on the xy -plane. Projecting to the xy -plane lets us talk of two different triangulations of the same polygon. In the dual view, we have two binary trees in the polygon whose leaves are the same set of points, one leaf in the center of each edge of the Hamiltonian circuit.

Thus our setting is either two triangulations of the same polygon, or two binary trees in a disk with the leaves in the boundary of disk and having the same leaf set.

This paper was motivated by the resemblance of the conclusions of Whitney's theorem to a group known as Thompson's group F . Elements of F are built from pairs of binary trees having the same number of leaves. The group F has been studied extensively since its introduction by R. J. Thompson around 1970.

The obvious resemblance was first noticed on the appearance of the expository article [23] by Robin Thomas. It was felt at the time that the proof of the 4CT, based as it was on hundreds of cases that needed to be checked by computer, could benefit from a little extra organization, and that this organization might just be supplied by the group structure of F . Unfortunately, first looks by several people found that the multiplication on the group and the colorings of the maps seemed to have absolutely nothing to do with one another.

In June of 2011, the authors of the current paper decided to take another look and found (with the help of the computer) that while the multiplication and colorings did not cooperate, there is a strong relationship between a presentation of F by generators and relations on the one hand, and machinery that attempts to create 4-colorings of maps in \mathfrak{M} on the other hand.

Because of our desire to introduce the group F , and since F is typically defined using pairs of binary trees, we will concentrate on pairs of trees in this paper. However, the triangulated polygons point of view has its uses and will be used when it is advantageous to do so.

1.1.4. Previous work. Starting with Whitney [26] in 1936, Whitney's theorem [25] has been used to formulate statements equivalent to the 4CT. Two from [26] are repeated as \mathbf{C}_{19} and \mathbf{C}_{20} in Section 5-5 of [19], along with statements due to several others. A later formulation by Kauffman [12] was discussed by R. Thomas in the [23]. A formulation [14] by Loday is more recent. Most work relevant to the current article started to appear after [12] and [23].

Given two triangulations of the same polygon, it has been known that there is a sequence of "diagonal flips" (terminology of [8]) that will sequentially modify one triangulation until it is identical to the other. Dually, given two finite binary trees with the same leaf set, there is a sequence of "rotations" (called transplantations in [13]) that will sequentially modify one tree until it is isomorphic to the other. Neither of these observations has the power to give a proof of the 4CT.

For trees, Kryuchkov [13] in 1992, and for triangulated polygons, Eliahou [8] in 1999 showed that if a sequence of modifications which preserves some extra structure could always be found, then the 4CT would follow. (Structure preserving moves for trees are called admissible transplantations in [13]. For polygons they are called signed diagonal flips in [8]. Details about the extra structures that must be preserved by the sequences of moves will be given later.) Specifically, if there is an appropriate structure that a sequence of moves preserves, then the sequence of moves leads to a desired coloring. This gives a pair of statements that imply the 4CT but that might be false. In [11], Gravier and Payan give an elegant proof that this is not the case and that these statements follow from the 4CT. There is actually more detail in the result of [11] and this will be discussed shortly.

Since the trees and polygons are finite, and the extra structures demanded by the results of [13] and [8] come from a finite set of possible structures, an exhaustive search will show if a given sequence of modifications satisfies the hypotheses of these papers. In [4], Carpentier gives a practical algorithm to decide if a given sequence of modifications has a corresponding structure that it preserves. The paper [4] uses triangulated polygons, but is easily recast to use trees.

Many of the reformulations of the 4CT using Whitney's theorem have an algebraic flavor. Eliahou and Lecouvey in [9] introduce the symmetric group and Loday in [14] introduces Lie algebras into discussion of coloring the maps in \mathfrak{M} .

The work of the papers discussed to this point is to establish equivalences with the 4CT. In [5], Cooper, Rowland and Zeilberger calculate proper, vertex 4-colorings for certain infinite sets of maps represented as pairs of trees. Decomposition into primes is also discussed. Their results are extended by Csar, Sengupta and Sukhompong in [6] to pairs of trees that lie in an interval in a certain lattice.

1.2. On the current paper. We consider maps in the class \mathfrak{M} and we translate the sequences of structure preserving moves of Eliahou and Kryuchkov into words in the generators of elements of Thompson's group F . We also derive a criterion that is structurally different from (but one assumes equivalent to) the criterion of Carpentier for a word in the generators to have an associated structure that is preserved by the word and thus lead to a coloring.

Implicit in this description is the possibility that there are words that cooperate with no structure and that lead to no coloring. This turns out to be the case and we call words that cooperate with some structure "successful." It is also possible that for a given map, different words that successfully lead to colorings lead to different colorings of the same map. Further, it is possible that a given word does not uniquely determine a coloring for a map.

A result and conjecture of ours relate to these remarks. We show that for prime maps, a successful word determines a unique coloring of the map (modulo permutations of the colors). We present an extension E of F that uses the same generators but fewer relations and conjecture that two successful words lead to the same coloring of a map if and only if they represent the same element of E .

We note that it is possible to give a well defined notion of a coloring of an element of F and it is elementary that the 4CT is equivalent to the statement that every element of F has a coloring. This ignores the specific representation of the elements of F as words in a certain generating set, and only asks that there be some word that succeeds. The results of [5] essentially discuss colorings of elements of F without reference to their representation as specific words in the generators.

1.2.1. *Subgroups and monoids.* There is a positive monoid P of F that is so named because F is the group of fractions of P in that every element of F is of the form pn^{-1} with p and n in P . We show that all elements in P have colorings and that the prime elements of P have unique colorings. This duplicates, possibly extends, and to some extent unifies several of the results in [5] and [6]. See Section 8.6.

An extra detail about the results of Kryuchkov and Eliahou is that structure preserving sequences of moves lead to 4-colorings that use all four colors. The converse by Gravier and Payan [11] assumes a coloring using all four colors and finds for that coloring a structure preserving sequence leading to the coloring. Thus any coloring using all four colors comes from some element of the group E .

Colorings using only three colors have no corresponding sequences. We refer to such colorings as rigid. (Non-rigid colorings are called flexible.) There is an easy characterization of rigid colorings in [11] and the well known fact that a map in S^2 can have a proper, face 3-coloring if and only if each face has an even number of edges is presented as Theorem 2-5 of [19]. The possibility of a 3-coloring is not a large problem since every map with more than 3 faces having a proper, face 3-coloring also has a proper, face 4-coloring that uses all four colors.

We show that the maps having 3-colorings correspond to a pleasantly described subgroup F_4 of F .

The subgroup F_4 corresponds to a certain “pattern” that assigns a fixed coloring to all finite binary trees. We call it the rigid pattern since it allows for no moves of the type considered by Kryuchkov. We briefly discuss what it takes for a pattern to lead to a subgroup of F and briefly consider the pattern that is the “least rigid.” This leads to a subgroup of F that seems not to have been considered before.

1.2.2. *Associahedra and color graphs.* Associahedra are convex polytopes whose vertices are finite binary trees or (dually) triangulated polygons. There is one associahedron in each dimension and we denote the associahedron of dimension d by A_d . The vertices of A_d are triangulated polygons having $d + 3$ edges or binary trees with $d + 3$ leaves (or $d + 2$ leaves if one leaf has been promoted to the title of “root” of the tree). Sequences of moves considered by Kryuchkov or Eliahou are walks along the edges of A_d with each move corresponding to a traversal of one edge.

Let us denote by \mathbf{v} a 4-coloring (with no requirement that it be proper) of a polygon with $d + 3$ edges. We can refer to \mathbf{v} as a “color cycle.” Then \mathbf{v} is either a proper, vertex 4-coloring or not for a given triangulation of the polygon depending on the triangulation. Thus certain vertices of the associahedron A_d can be said to be valid for \mathbf{v} and certain vertices are not. The structure preserving moves are the ones corresponding to edges of A_d in the subgraph of the 1-skeleton of A_d spanned by the vertices that are valid for \mathbf{v} .

This makes the subgraph spanned by the vertices that are valid for \mathbf{v} an interesting subgraph to look at. We refer to it as the color graph of \mathbf{v} . It follows from the results and discussion in [11] that a color graph is either connected or has no edges.

The diameter of A_d is well understood. From [17], we know that the diameter is $2d - 4$ for $d \geq 10$. The diameter of the color graphs of the various \mathbf{v} can be considerably higher, and we show examples where the diameter is $\lfloor d^2/4 \rfloor$. Interestingly, the examples involve pairs of vertices of the associahedron considered in [5].

We call the complement of the color graph of \mathbf{v} the zero set of \mathbf{v} for reasons that we will explain later. We raise questions about the relative sizes and structures of a color graph and its complementary zero set.

We present examples that show that the structure of the color graphs can be rather complicated, independent of the examples of large diameter. A prime map with a rigid coloring represents two isolated vertices in a color graph. However, if a rigidly colored prime map is combined with a flexibly colored prime map, then the coloring of the resulting non-prime map is flexible. The “dissolving” of the rigidity of the rigid factor by the presence of the flexible factor can be very inefficient and we give an example of how this takes place.

1.2.3. Enumeration. The fact that we consider all proper, 4-colorings gives us an opportunity to try to count them. It is known that counting the colorings of a map is $\#P$ -complete [24], but that does not make counting them impossible. We give examples of maps with very large numbers of colorings and ask whether these maps have the largest numbers of colorings for maps of a given size. One question has been since answered by P. D. Seymour [20]. See the note after Question 15.8.

We also characterize certain structures and this allows us to count them as well.

A proposition in [5] gives properties of any color cycle \mathbf{v} that has a non-empty color graph. We supply a converse to this proposition and use it to count those color cycles with non-empty color graphs.

We can also characterize those color cycles that have non-empty color graphs but which are rigid. This lets us count such color cycles and thus the complementary set of color cycles that are flexible.

Many questions about enumeration are raised.

1.2.4. Other. It is well known that coloring behavior changes as the genus of the surface containing a map goes up. The effects of this on move sequences is discussed in [8], and we give a brief discussion on its effects on the current paper’s point of view. In particular, there is a larger group of Thompson’s known as V and we show that there is no counterpart for V of the statement that “all elements of F have a coloring.” Interestingly, the standard example on the torus (an embedding of K_7) does not supply an example, but the standard example on the projective plane (an embedding of K_6) does.

1.3. In the paper. We describe the flow of the paper and add a bit of detail to the above list.

Section 2 establishes definitions and notation.

Trees are the central objects of the paper and all definitions of trees and related structures are given in Section 3. The machinery needed to define the faces of the associahedra are introduced. The action of the dihedral groups on trees is defined.

The associahedra are introduced in Section 4. Vertices are binary trees, and rotations are defined as a move along one edge and thought of as a modification of the tree at one end of the edge to make it look like the tree at the other end.

Color (not mentioned in the sections before) is added in Section 5. Since we view maps as made of pairs of trees, there is extensive discussion of colorings of trees and tree pairs. The easy consequence of Whitney’s theorem that the 4CT is equivalent to the colorability of pairs of trees is pointed out. The orientation of colors used around a single vertex (called its sign) is introduced and discussed.

Section 6 combines color with rotations. The use of paths along edges to find colorings is introduced and we restate the conjectures of [13] and [8] to read that every pair of vertices in an associahedron has an edge path between them that succeeds in carrying a coloring from the first vertex to the second. This conjecture implies the 4CT, and the converse follows from the main result of [11]. We introduce the rigid colorings that are carried by no edge path. These correspond to triangulations of the 2-sphere that have proper, vertex 3-colorings. We also introduce the notion of a color graph on an associahedron. Such a graph has as vertices all trees colored by a single “vector” of colors and the edges are all the rotations between the vertices that stay within the colorings given by the vector. By the result of [11], this graph is either connected or has no edges.

Thompson’s group F and its extension E are described in Section 7. Algebra is combined with rotations to represent elements of F as edge paths in an associahedron.

Section 8 goes as far as it can combining color with the multiplication on Thompson’s group. The equivalence of the 4CT to the colorability of tree pairs is given the immediate translation into the equivalence of the 4CT to the colorability of elements of F . Since F is finitely generated, this can be reformulated as an equivalence, Theorem 8.2, that sounds simpler but that adds little true knowledge. A compatibility lemma is given that finds cooperation between the multiplication and colorings in certain situations and is exploited to show that all elements in a certain submonoid of F have colorings and that some of them have unique colorings. These results have other proofs. They also unify a number of the results in [5].

Section 9 covers thoroughly the uncooperative 3-colorings. These correspond to a nice subgroup of F , are easy to characterize, and thus, in Section 15, easy to count.

Section 10 further considers color graphs and also looks at their complements. We show that color graphs (which live in the graph of edges of associahedra) can be more contorted in appearance than the full edge graph of the associahedron that they live in.

Section 11 develops the criterion that detects the ability of an edge path to successfully drag a coloring from one tree to another. It is similar in spirit, but different in detail from the criterion in [4]. This test for success builds, from the edge path, a signed graph which will be balanced if and only the edge path gives a coloring. Prime maps have connected signed graphs, and (up to permuting the colors) a successful edge path leads to a unique coloring of the prime map. Colorings of non-prime graphs are combinations of colorings of prime graphs. The combination of a face 4-coloring using four colors on one prime factor with a face 3-coloring on another prime factor results in a coloring that can still be created by an edge path in spite of the fact that the face 3-coloring could not have been so created on its own. When this happens, the signed graph constructed from the path has components that correspond to more than one prime factor. Restatements of earlier conjectures are given and the discussion is given that brings the group E into the picture. Conjectures about the relevance of the group E are given.

The remaining sections cover material that is not part of the main flow of the paper.

Section 12 separates those color arrangements that color no trees from those that color some trees by characterizing the latter. (Except for miniscule trees, there are

no arrangements that color all trees.) This supplies a converse to a proposition in [5].

In the discussion about face 3-colorings, a pattern for coloring all trees comes up. This does not automatically color tree pairs, since a coloring of a tree pair has a condition to be met (colors must agree on the leaves of the two trees). The pattern considered in Section 9 produces face 3-colorings when a match occurs. This pattern also leads to the subgroup F_4 of F that we mentioned above. Section 13 briefly looks, in some generality, at conditions that make a pattern lead to subgroups of F . Section 13.2 looks at a particularly simple pattern that seems to lead to a not so simple subgroup of F . Section 13.3 demonstrates the limit of usefulness of the pattern of Section 13.2 to the 4CT.

Section 14 takes a brief look at standard examples of maps on the torus and projective plane from the point of view of the current paper and relates the examples to another of Thompson's groups known as V . As expected, the situation is quite different. In particular, the obvious generalization to V of the statement that all elements of F have a coloring is false.

Section 15 gathers all our results and questions about counts. Formulas are given that count the colorings that are valid for at least one tree, that count colorings that lead to face 3-colorings of maps, and that count colorings that lead to "true" face 4-colorings of maps (all four colors are used). From our computer calculations, we also noticed that certain maps have many more colorings than others. One map in particular (the biwheel from the dual point of view) seemed to have approximately twice as many as any other. We raise the question of which maps of a given size have the most colorings and have candidates for the top four counts of colorings for a given size. Our candidate for the top count and the fact that it corresponds precisely to the biwheel has been since proven correct by P. D. Seymour [20]. The section ends with other questions of enumeration motivated by other parts of the paper.

Section 16 raises questions that fit nowhere else.

1.4. Thanks. The authors extend their thanks to Jeff Nye for all of his help and ideas with the programming.

2. STARTING FROM WHITNEY'S THEOREM

We establish some notation to accompany the setting described in Section 1.1.3. Certain terms are defined carefully. However, others are to be interpreted intuitively by the reader during this section and they will be defined more carefully in the remaining sections of the paper.

2.1. The equator. For a map M in \mathfrak{W} we know the dual graph of M in S^2 has a Hamiltonian circuit E that we will call the equator. We can take this circuit to be the circle $z = 0$ in the 2-sphere $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. We can also insist that some point of the intersection of M with E is the point $(0, 1, 0)$. From now on we will insist that an $M \in \mathfrak{W}$ is arranged in this way on the 2-sphere.

We call the points in S^2 with $z \geq 0$, the northern hemisphere, and the points with $z \leq 0$ the southern hemisphere. From Section 1.1.3, we know that the intersection of M with each hemisphere is a binary tree.

For an $M \in \mathfrak{W}$, we will usually use D to denote the intersection of M with the northern hemisphere, and R to denote the intersection of M with the southern hemisphere. The reason for these letters will be clarified in Section 7.1.

If the dual graph of M in S^2 has no parallel edges, then we follow Loday in [14] and call M *prime*. Such maps are called irreducible in [5].

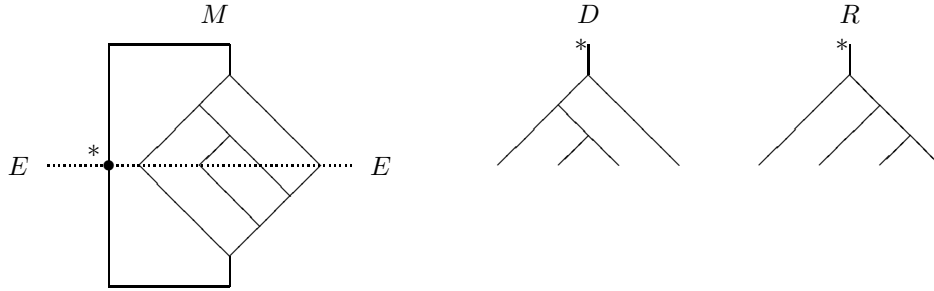
2.2. Rooting and orienting the trees. Given $M \in \mathfrak{W}$ with D and R as above, we declare that the common root of D and R is the point $(0, 1, 0)$. The leaves, which are common to both D and R , are all other points in M on the equator. Note that in each of D and R , the root has degree 1 and the root and leaves together account for all the vertices of degree 1.

We then embed each of D and R (separately) in the xy -plane by vertical projection. If the fact that the two images intersect is bothersome, then D can be moved two units in the negative x direction, and R can be moved two units in the positive x direction. The purpose of the projection is to discuss orientation in a well defined way.

With D and R projected to the xy -plane, we associate to each vertex v of degree 3 a cyclic order on the edges that impinge on v by using the counterclockwise order on the edges as viewed from above the plane. We also get a linear order on the leaves. The order is obtained from a counterclockwise walk (as viewed from above) that starts and ends at the root around the circle of radius 1 that contains the root and leaves.

Since the vertices that are neither root nor leaf are all of degree 3, we can call each of D and R a binary tree. Thus we have a process that takes each $M \in \mathfrak{W}$ to a pair (D, R) of finite, rooted, oriented, binary trees that have the same number of leaves. We will later be very careful with definitions, but the reader can infer enough definitions at this point for us to continue.

Below left is a map M with equator (dotted line) labeled E and chosen root labeled $*$. To the right are the two trees D and R obtained from M .



Note that a pair of trees (D, R) that comes from a map $M \in \mathfrak{W}$ comes with a one-to-one correspondence from the leaves of D to the leaves of R that respects the linear orders on the leaves. If D and R are regarded as subsets of M (or the embedding of M in S^2), then this one-to-one correspondence is just the identity map. If this correspondence is extended to the roots, then we also have a one-to-one correspondence from the vertices of degree 1 in D to the corresponding set in R . We will refer to these correspondences often.

From now on a pair of finite trees (D, R) will always carry the assumption that the two trees have the same number of leaves. We will remind the reader of this from time to time.

2.3. Reversing the process. Conversely, we can take a pair (D, R) of finite, rooted, oriented, binary trees with the same number of leaves and create a map in \mathfrak{W} . The method of creating a map should be clear. We will sketch the steps, and definitions made later will fill in details.

The orientations of the vertices induce a cyclic order of the vertices of degree 1 (all leaves and the root), which becomes a linear order if the root is declared to come first.

We map D to the northern hemisphere of S^2 and R to the southern hemisphere so that only the leaves and roots end up on the equator, so that the two roots go to the point $(0, 1, 0)$, so that the leaves of D occupy the same points as the leaves of R , and so that a counterclockwise walk (viewed from above) around the equator starting at the root visits the leaves of the two trees in an order that agrees with the order induced from the orientations.

Note that the number of faces is equal to the number of vertices on either D or R of degree 1.

The failure to always get a dual with no parallel edges is more easily illustrated than discussed. The two trees shown below will produce a map that fails this property. Recall that D will live on the northern hemisphere so that its vertical projection to the xy -plane looks as shown below and that R will live in the southern hemisphere so that its vertical projection to the xy -plane looks as shown below.



We follow [14] again and say that a pair of trees is *prime* if the process above produces a map whose dual has no parallel edges.

It is just as easy to build a pair of trees that creates a map that has a separating circuit of length 3. We leave this for the reader.

The working class of maps for the rest of this paper are the maps in the class \mathfrak{W} .

2.4. Triangulations. Dual to the pairs of trees are the triangulated polygons described in Section 1.1.3. The equator E breaks the triangulation dual to a map $M \in \mathfrak{W}$ into two triangulated polygons whose common boundary is E .

This dual view works with pairs (P_1, P_2) for which the P_i are triangulated polygons with the same number of boundary edges. Each P_i is triangulated using no vertices other than the vertices of the polygon, and there is a cyclic order preserving bijection between the vertices of the two polygons. The number of edges in the boundary of each P_i is the number of faces of the map M . This view makes it clear that the dihedral group of order $2n$ acts on the set of maps in \mathfrak{W} with n faces. See Section 3.8.

3. TREES

We do a lot with trees, and we use this section to define the terms and structures that we need. Some of the oddness in our definitions is motivated by the discussion in Section 2.

3.1. General trees. A *tree* is a connected graph with no circuits. A finite tree is a tree with finitely many vertices. A locally finite tree is a tree in which the degree of every vertex is finite.

Every tree must have a root, and we will be very restrictive about where the root can be. For us, the root of a tree must have degree 1. We will emphasize this by saying that such a tree is *rooted*.

Each vertex in a tree has a distance to the root defined as the length (in edges) of the unique simple walk from the vertex to the root. If v is a vertex in a tree with a root, then the *children* of v are those vertices adjacent to v that are farther from the root than v is. The vertex adjacent to the root is the only child of the root, and we insist that the root have a child. The *leaves* of a tree are the vertices with no children. The *trivial tree* is the unique tree with one leaf.

The inverse of “child” is “parent.” Every vertex except the root has a unique parent, and the root has no parent. The transitive closure of “child” is “descendant” and the transitive closure of “parent” is “ancestor.” The root is an ancestor of all other vertices in a tree.

In a *binary tree* every vertex that is neither root nor leaf has degree 3.

If T is a tree, then the *internal vertices* of T are the vertices of degree greater than 1. The *internal edges* are the edges where both endpoints are internal vertices. While not standard terminology, we need to call the edges that are not internal edges the *external edges*. It will be convenient to separate the external edges into the *root edge*, the edge that impinges on the root, and the *leaf edges*, those edges that impinge on leaves.

The root and the root edge are motivated by Section 2.2, and are there for sometimes technical, sometimes formal, and sometimes practical reasons. Since the root and root edge must always be there, it adds no information in a drawing to include them. So the occasional drawing of a tree with no root and root edge shown will be assumed to be “augmented” by these items.

The root edge will be inconvenient at times. However it would be inconvenient more often if it were omitted.

3.2. Orders in trees. A tree is *locally ordered* if for every vertex v there is a cyclic order defined on the set of edges that impinge on v . The cyclic order on the edges gives a cyclic order on the vertices adjacent to v . For any vertex v other than the root, this leads to a linear order on the children of v by starting the linear order with the vertex that follows the parent in the cyclic order. The children of the root have a linear order since the number of children of the root is 1.

If a tree T is embedded in the plane, then the embedding induces a local order by using the counterclockwise cyclic order of the edges that impinge on a vertex v as described in Section 2.2.

Conversely, every finite, ordered tree can be embedded in the plane so that the local order induced from the embedding agrees with the given order.

An embedding can be referred to as a drawing. We will draw trees with the roots at the top. See the paragraphs at the bottom of Page 8 and top of Page 9 in [15] for a discussion of this choice.

An isomorphism of locally ordered trees is required to respect both root and local order. It follows that the only automorphism of a locally ordered tree is the identity.

A locally ordered tree can be given the prefix total order on all the vertices. This is the unique linear order that restricts to the local linear order on the children of each vertex and that for two children v and w of a single vertex, if $v < w$, then the descendants of v precede w which in turn precedes the descendants of w . The restriction of this total order to the leaves gives a total order on the leaves.

In a locally ordered binary tree, every vertex with two children has the first child designated as the left child and the other child is the right child. A locally ordered binary tree can also be given the infix total order. This is the unique linear order that restricts to the local linear on the children and that places each vertex after its left child and the descendants of the left child and before its right child and the descendants of the right child. The restriction of the infix order to the leaves is the same as the restriction of the prefix order to the leaves.

The induced order on the leaves (from prefix or infix order) will usually be referred to as the left-right order on the leaves since, with the tree drawn with the root at the top, visiting the leaves moving left to right in the drawing follows the order.

We will often have need to refer to the i -th leaf of a finite, locally ordered tree, and this will refer to the i -th leaf in the left-right order of the leaves, with the count starting from 1.

3.3. Standing assumptions on trees. In the rest of this paper, all trees will be locally finite, rooted, locally ordered, and no vertex will have degree 2. These assumptions will not be repeated in statements.

3.4. Subtrees. A subtree S of a tree T is a subgraph that is also a tree and that satisfies the extra condition that if v is a vertex of S other than the root of S , then either all the children of v in T are in S , or none of the children of v in T are in S . The root of S must have only one child.

A local order on a tree is inherited by a subtree.

Since we assume T has a root, there is a unique vertex in S that is closest to the root of T and we will insist that this vertex be the root of S .

3.5. Standard model for binary trees. Binary trees can be given a very standard structure that makes them all subtrees of a common structure. This will be convenient in many places.

The vertex set of the infinite, locally ordered, binary tree \mathcal{T} consists of a single special element that we will denote $*$, together with the set $\{0, 1\}^*$ of all finite words (including the empty word \emptyset) in the alphabet 0 and 1. In $\{0, 1\}^*$, we take *concatenation*, *prefix*, *suffix* to have their usual meanings. Concatenation will be denoted by juxtaposition. The left child of a vertex u in $\{0, 1\}^*$ will be $u0$ and the right child of u will be $u1$. The only child of $*$ will be \emptyset . Words like *parent*, *ancestor*, *child*, *descendant* will have their usual meanings. We order the children of $u \in \{0, 1\}^*$ by $u0 < u1$. The edges of \mathcal{T} are all connections from parent to child.

Declaring $*$ to be the root makes \mathcal{T} a rooted, locally ordered, binary tree. The infix order will be used on \mathcal{T} at times.

Throughout the paper we will need to discuss words. We will use notation similar to regular expressions and write 0^n for a string of n zeros, 0^+ for a non-empty finite string of zeros and 0^* for a possibly empty finite string of zeros. Thus $1^+0^+1^*$ represents at least one 1 followed by at least one zero followed by zero or more ones. The expression (abc) refers to the string abc , and $(abc)^n$, $(abc)^*$, $(abc)^+$ refer, respectively, to n appearances, zero or more appearances, and one or more appearances of the string abc . Lastly $[xyz]$ refers to a choice of one of x , or y , or z , so that $[xyz]^n$, $[xyz]^*$, $[xyz]^+$ refer, respectively, to n , zero or more, one or more appearances of independent choices from x , or y , or z .

Recall that all trees are locally ordered. It is standard that any finite, binary tree can be realized uniquely as a finite subtree of \mathcal{T} so that its root is $*$, so that its vertex set is closed under the taking of prefixes and for which $u0$ is in T if and only if $u1$ is in T . It will be convenient to view all finite, binary trees this way.

Each $v \in \{0,1\}^*$ is a vertex of \mathcal{T} and of any tree in \mathcal{T} that contains v . As a word, it also contains the information on how to get to v as a walk from \emptyset . As such it can be thought of as an “address.” This makes for the odd statement that the word v is the address for the vertex v , but at times this will be convenient.

Subtrees are defined as for subtrees of general trees. There are some special notations for certain subtrees.

If T is a binary tree and v is a vertex in T with parent w , then T_v will denote the subtree $\{vu \mid vu \in T\} \cup \{w\}$. The root of T_v is w and every leaf of T_v is a leaf of T . If T is a binary tree, then T_0 will be referred to as the *left subtree* of T and T_1 will be referred to as the *right subtree* of T . We will be consistent with the words “tree” and “subtree” and insist that a finite, binary tree in \mathcal{T} have $*$ as its root, while subtrees can have any vertex in \mathcal{T} as its root.

The advantage of putting finite binary trees in \mathcal{T} is that if T and S are binary trees, then so are $S \cap T$ and $S \cup T$. Recall that trees are rooted at $*$.

3.6. Inducting with trees. We will frequently appeal to either of two inductive arguments without giving details.

Finite trees are well founded (every set has a minimal element where vertices of a tree are partially ordered by declaring descendants less than ancestors) and as such can have constructions or statements about them argued by well founded (Noetherian) induction which lets a statement or construction be applied to a vertex if it applies to all of its descendants. We will often simply say that something can be defined or argued inductively. For example, *height* can be defined inductively by declaring the leaves of a finite tree to have height zero, and any vertex whose descendants all have defined heights is defined to have height one greater than the largest height of its children.

Another argument is based on a binary operation on binary trees. Given two binary trees, S and T , there is a unique binary tree $S \hat{\ } T$ whose left subtree is isomorphic to S and whose right subtree is isomorphic to T . Every binary tree that has more than one leaf is uniquely of the form $S \hat{\ } T$. Since $S \hat{\ } T$ has more vertices than either of S or T , we can induct on the size of a binary tree.

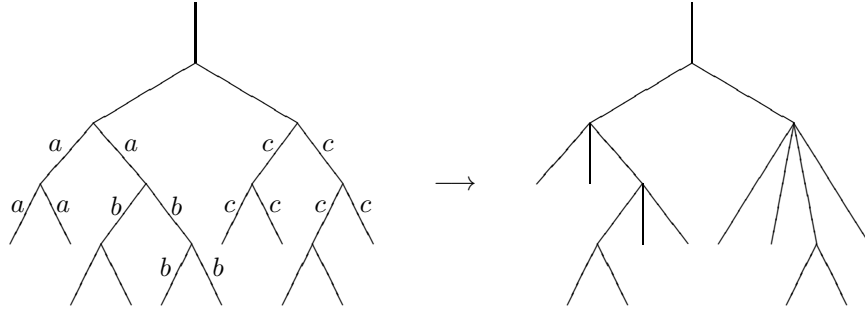
3.7. Projections. Since we will study pairs of binary trees, we will be comparing structures of trees. The following will help organize trees into related groups of various sizes.

If T is a finite, binary tree then a projection of T can be defined based on certain data. Here is one place where the root edge annoys. We say that two subtrees S_1 and S_2 of T are *edge disjoint* if removing the root and root edge from each of S_1 and S_2 leaves them with no edge in common.

The data for the projection consists of a finite set T_1, \dots, T_n of subtrees in T , which are pairwise edge disjoint. To avoid trivialities, we require that each T_i have at least three leaves.

We form the *projection determined by* T_1, \dots, T_n as follows. For each i with $1 \leq i \leq n$, let w_i be the child of the root of T_i . We remove from T all the non-root edges of T_i and all the internal vertices of T_i except for w_i . Then for each leaf of T_i , we add an edge that connects that leaf to w_i .

We show an example below.



In the illustration above, three subtrees are used. Each of the three subtrees is pictured in the tree on the left by having all of its non-root edges given a common label. Edges labeled a are of a subtree with one internal edge and three leaves. Edges labeled b are of a similar subtree (but not isomorphic as an ordered subtree). The edges labeled c are of a subtree with two internal edges and four leaves. The subtree with edges labeled a is edge disjoint from the subtree with edges labeled b because of the technicalities in the definition of edge disjoint.

Note that a projection determined by subtrees T_1, \dots, T_n can be thought of as the result of shrinking all the internal edges of the T_i to have length zero.

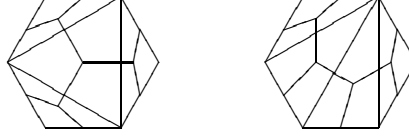
A projection inherits a local order from the infix order of the original tree.

Note that if T is any finite tree, then there is a finite set of binary trees that have T as a projection. This will be used in Section 4.

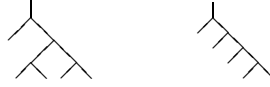
3.8. Root shifts and reflections. An operation on trees is easiest to describe using the dual view. A rooted tree embedded in the plane is naturally dual to a triangulated polygon. A finite binary tree T with n external vertices ($n - 1$ leaves and one root) can be embedded in a regular, triangulated n -gon P in the plane so that it is dual to the triangulation, so that the local order is preserved, and with one external vertex of T in the interior of each edge of P . We also insist that a horizontal edge in the polygon is at the top and that the root of T is in this top edge. Given P and T as described, we say that P is the triangulated n -gon dual to T and that T is the tree dual to the triangulated n -gon P .

Below are two examples

(2)



The trees corresponding to these examples are drawn below in the usual way.

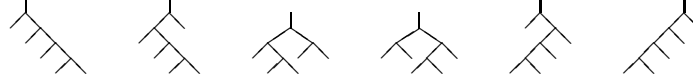


If symmetries of the n -gon P are now applied and we adhere to the convention that the top edge contains the root of the tree, then we get new trees that are isomorphic as unrooted, unordered trees to the original tree T but which might or might not be isomorphic as rooted, ordered trees. If all possible symmetries are applied to the left figure in (2), then the set of trees obtained this way contains only two trees. These are shown below.

(3)



However if all possible symmetries are applied to the right figure in (2), then the set of tree obtained is larger and is shown below.



Note that because of the symmetries present in the examples chosen, all of the above modifications could have been accomplished by a rotation of the n -gon P . This is not always the case.

We use the term *root shift* to refer to the modification of a finite, binary tree T obtained by a rotation of its dual triangulated n -gon. We do not use the term rotation since the word rotation will be used for completely different modification of a tree. An alteration of a finite, binary tree obtained by an orientation reversing symmetry of its dual n -gon will be called a *reflection*.

4. THE ASSOCIAHEDRA

The associahedra are cellular complexes whose vertices are finite, binary trees. There is one associahedron in each dimension. The vertices of the d -dimensional associahedron are all the trees with d internal edges, $d + 1$ internal nodes, $d + 2$ leaves and pairs of such vertices (trees) create maps with $d + 3$ faces. Thus there are several integer values associated to one associahedron and will require care to keep track of. The books [21] and [15] refer to the d -dimensional associahedron by K_{d+2} so their notational parameter (subscript) is the number of leaves.

We will denote the d -dimensional associahedron by A_d .

The number of vertices in A_d is number of trees with $n = d + 1$ internal nodes which is

$$C(n) = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$$

the n -th Catalan number. This is sequence A000108 in the Online Encyclopedia of Integer Sequences. It will be relevant that the order of growth of this sequence is 4^n . This is revisited in Section 15.1.

The associahedra are also convex polytopes. Their role as polytopes is not needed here. However the notions of faces, especially edges (faces of dimension one), will be particularly important. We will define faces formally without justifying the use of the word face. That the A_d are also cubical complexes does not concern us.

4.1. The faces. A face in A_d is determined by a projection (Section 3.7) of a vertex of A_d (finite binary tree with $d + 2$ leaves) and each isomorphism type of a projection gives a face. Isomorphisms of projections are required to preserve local order. The face determined by a projection S is the set of all vertices of A_d that have a projection isomorphic (as ordered trees) to S .

The dimension of the face is the total number of internal edges in the selected subtrees of the projection.

The structure of a face is a product of various A_i . If T is a vertex in a face, then there are selected subtrees T_1, T_2, \dots, T_n of T that specify the projection that defines the face. Any other vertex T' in the same face can be obtained from T by replacing each T_i by a binary subtree with the same number k_i of leaves as T_i . For each i , this “space of replacements” gives an isomorphic copy of A_{k_i-2} . The structure of the face is the product

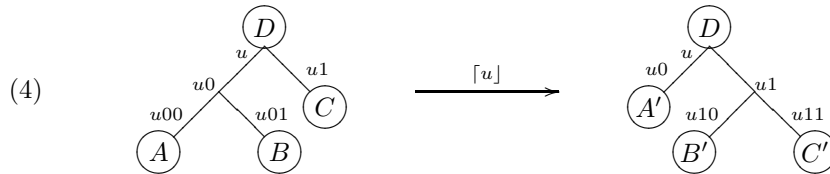
$$A_{k_1-2} \times A_{k_2-2} \times \cdots \times A_{k_n-2}.$$

Note that if S is a projection determining a face (namely S is a locally ordered, rooted, finite tree), then the dimension of the face determined by S is the sum over the internal vertices v of S of the quantity $d(v) - 3$ where $d(v)$ is the degree of v .

In the projection illustrated in Section 3.7, the tree on the left is a vertex in A_9 , and the face determined by the projection shown has the structure $A_1 \times A_1 \times A_2$.

4.2. Edges and rotations. The simplest non-trivial faces of A_d are the edges. An edge with vertex T at one end is determined by a subtree T_1 of T having one internal edge and three leaves. There are exactly two binary trees having three leaves and the other vertex T' of the edge is obtained from T by removing T_1 and replacing it by “the other” tree with three leaves.

Two trees that form the vertices of an edge are shown below. The arrow labeled $\lceil u \rceil$ will be explained in the paragraphs below. Recall that we view all finite binary trees as living in \mathcal{T} .



In each figure, the edges shown are the all the edges in the selected subtree of three leaves that specify the relevant projection. The circles shown contain the rest of the tree. The vertices are labeled with their addresses.

The letters in the circles indicate corresponding subtrees in the two trees. The root edge is an annoyance here. For example, the circle containing A indicates a subtree whose root is $u0$. However, it is more convenient to leave the root edge outside the circle A in the figure. The subtree D is literally the same subtree in

both trees and contains the root. The subtrees A and A' are isomorphic but not identical since they do not use exactly the same vertex set. Similar comments apply to B and B' as well as C and C' .

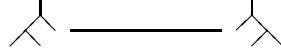
We refer to the transition from one of the trees shown to the other as a *rotation*. Thus traveling along an edge in an associahedron is performing a rotation of one tree into another. Our rotations are the transplantations of [13] and are dual to the (unsigned) diagonal flips of [8].

The particular rotation illustrated above will be referred to as the rotation of the left tree (or from the left tree to the right tree) at u and will be denoted $\lceil u \rceil$. The fact that trees with many different shapes might use the vertex with address u will not be a problem, and in fact the common notation for all rotations using a subtree with root at u will be a benefit.

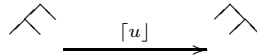
We invent the notation \bar{u} formally so that $\lceil \bar{u} \rceil = \lceil u \rceil^{-1}$ is the rotation from the right tree to the left tree. We can also abuse the notation a bit and write $\lceil v \rceil$ to mean a rotation in either direction depending on whether v is of the “form” u or \bar{u} , so that $\lceil v \rceil = \lceil u \rceil^{-1}$ if $v = \bar{u}$.

The vertices u and $u0$ in the left tree in (4) will be called the *pivot vertices* of the rotation $\lceil u \rceil$ shown there. The vertices u and $u1$ in the right tree in (4) will be the pivot vertices of the rotation $\lceil \bar{u} \rceil$.

4.3. Examples. The simplest non-trivial associahedron is A_1 and is a single edge shown below.

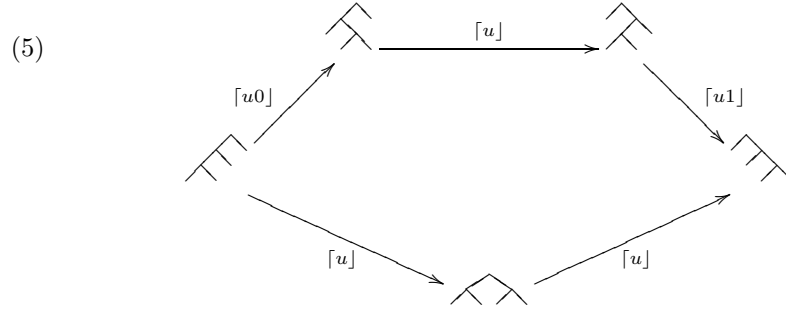


We can embed the trees in the figure above into larger trees so as to recover the figure (4). This would map the vertex \emptyset in the trees above to the vertex u in the trees in (4). If we think of the figure above as a simplification of (4) by forgetting to put in the trees A, A', B, B', C, C' and D , then the figure above can represent an arbitrary edge (1-dimensional face) in some A_d . With \emptyset mapped to a vertex u , in (4), we can redraw the figure above as follows (with the root edge omitted for simplicity) to show such an edge.



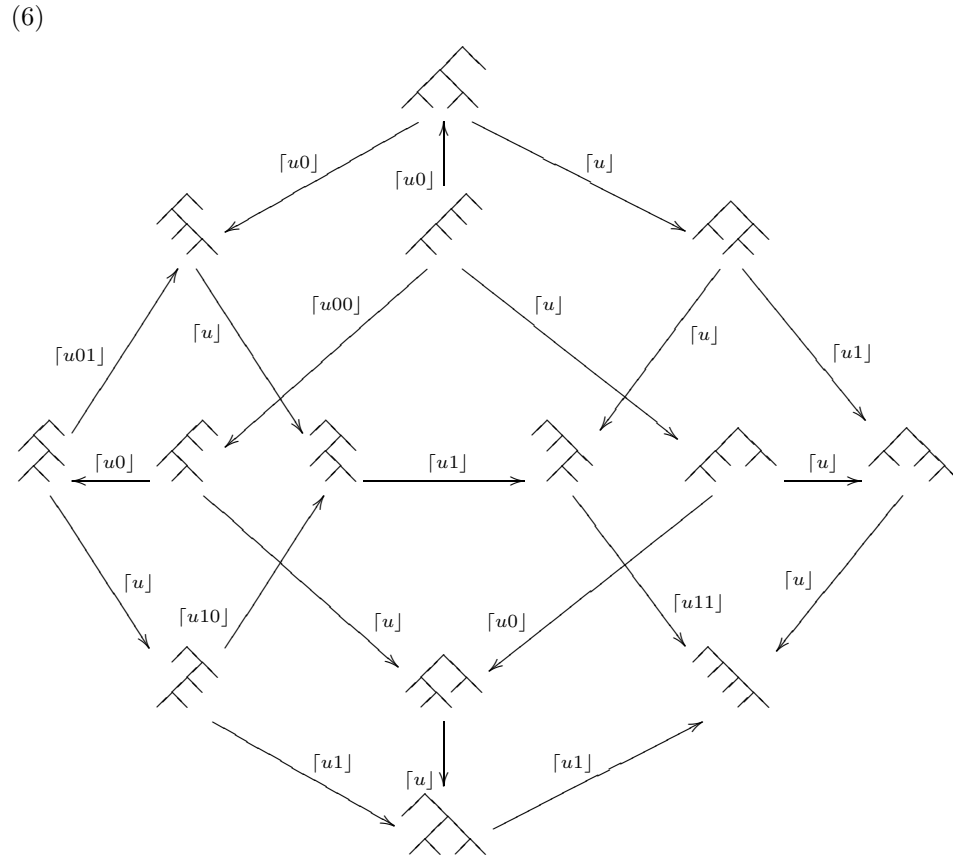
This does not give a unique edge in a higher dimensional associahedron since the rotation $\lceil u \rceil$ might apply to many vertices in an associahedron. For example, the rotation $\lceil \emptyset \rceil$ applies to any tree (vertex of an associahedron) whose left subtree is not trivial.

In a similar spirit, we show a typical 2-dimensional face that is isomorphic to A_2 below.





In the figure above, u is the address of the top vertex drawn in each of the subtrees shown.

There can also be 2-dimensional faces of the form $A_1 \times A_1$. There are three of these in the figure below which shows a typical 3-dimensional face of the form A_3 .



In the figure above, there are six 2-dimensional faces of the form A_2 .

Recall that each face corresponds to a projection. The central square face corresponds to the tree  and the tree  corresponds to the top central pentagonal face.

4.4. Symmetries. Note that the dihedral group of order $2(d+3)$ acts on the associahedron A_d by root shifts and reflections on the individual trees. See Section 3.8. That the action preserves the edges of A_d is easy to see from the dual view of triangulated polygons. In that setting an edge in A_d corresponds to an unsigned diagonal flip of [8]. This replaces two triangles in a triangulated n -gon that share an edge by “the other two triangles” with the same union as the original two triangles. The reader can check that the two examples in (2) are connected by an edge in A_3 .

5. COLOR

5.1. The colors. All our face and edge colors will come from $\mathbf{Z}_2 \times \mathbf{Z}_2$. We use $\mathbf{Z}_2 \times \mathbf{Z}_2$ since addition and subtraction are identical in this group. Notations such as $(0,0)$ and $(0,1)$ and so forth are cumbersome, and we will “code” the elements of $\mathbf{Z}_2 \times \mathbf{Z}_2$ by the mapping $(0,0) \rightarrow 0$, $(0,1) \rightarrow 1$, $(1,0) \rightarrow 2$ and $(1,1) \rightarrow 3$. In other words, the 0 and 1 in each (x,y) is thought of as a binary digit. In spite of this coding, the view that the colors reside in $\mathbf{Z}_2 \times \mathbf{Z}_2$ will be retained since we will use the arithmetic in $\mathbf{Z}_2 \times \mathbf{Z}_2$ to add colors. Thus 0 is the identity, we have $1+2=3$, $1+3=2$, $2+3=1$, and we have $n+n=0$ for any $n \in \{0,1,2,3\}$.

5.2. Face and edge colorings. It is standard that for planar cubic maps, a proper, face 4-coloring exists if and only if there is a corresponding proper, edge 3-coloring. This observation goes back to Tait [22]. One uses colors from a group of four elements (such as $\mathbf{Z}_2 \times \mathbf{Z}_2$). Given a proper, face 4-coloring, one colors the edges with the difference of the colors of the two faces that impinge on the edge. The identity is never the color of an edge. That the result is a proper, edge 3-coloring is a trivial exercise. The reverse direction depends on the planarity of the map. See Theorem 4-3 of [19]. The planarity is essential. This is discussed in Section 14.1.

Now let M be in \mathfrak{W} with (D,R) the associated pair of trees. Recall the one-to-one correspondence from the vertices of degree 1 in D to the similar set in R . Since we are going to discuss edges, we can also think of this correspondence as between the set of external edges in D to the set of external edges in R .

If M has a proper, edge 3-coloring, then the coloring restricts to a proper, edge 3-coloring of both D and R . The important feature of these colorings of D and R is that they give the same color to corresponding external edges of D and R . This leads to the next discussion.

5.3. Coloring binary trees. Let T be a finite, binary tree with n leaves. A *color vector* for T is an n -tuple $\mathbf{c} = (c_1, \dots, c_n)$ with values in $\mathbf{Z}_2 \times \mathbf{Z}_2$. The vector \mathbf{c} is thought of as an assignment of colors to the edges impinging on the leaves of T . The left-right order of the leaves of T is used to number the leaves from left to right starting at 1, and the color c_i is assigned to the edge impinging on the i -th leaf of T .

We also think of the color c_i as being assigned to the i -th leaf itself as well as the edge impinging on it. In fact, it will often be convenient to think of a color assigned to an edge in a tree as also being assigned to the vertex at the lower end of the edge. As a special case, a color assigned to the root edge is not only thought of as assigned to \emptyset but also to $*$. We call this color the root color.

One inductively argues the following.

Lemma 5.1. *Given a finite, binary tree T and color vector \mathbf{c} for T , there is a unique coloring of the edges of T from $\mathbf{Z}_2 \times \mathbf{Z}_2$ so that at every vertex v of degree 3 in T the sum of the colors of the edges impinging on v is zero.*

We refer to the coloring of T given in Lemma 5.1 as the coloring determined by the color vector \mathbf{c} .

Note that the coloring in Lemma 5.1 is not guaranteed to be a proper, edge 3-coloring for several reasons. First, we did not restrict the colors in \mathbf{c} to the non identity elements of $\mathbf{Z}_2 \times \mathbf{Z}_2$. Second, even if we made such a restriction, there is nothing to prevent the identity from showing up as the computed color of an edge not ending at a leaf. Of course if this happens, then the colors leading to the computed identity will also be identical.

If the identity is used in \mathbf{c} , and the tree has more than one leaf, then at least one vertex will not have three different colors for the edges ending at that vertex. A last comment is that if no edge impinging on a given vertex v of degree 3 has color 0, then because of the arithmetic in $\mathbf{Z}_2 \times \mathbf{Z}_2$, the three edges impinging on v have different colors. We have argued the following.

Lemma 5.2. *If T is a finite, binary tree with at least two leaves and \mathbf{c} is a color vector for T , then the coloring of T determined by \mathbf{c} will be a proper, edge 3-coloring of T if and only if the identity in $\mathbf{Z}_2 \times \mathbf{Z}_2$ is not used in the coloring of T determined by \mathbf{c} .*

We say that a color vector for a binary tree is *valid* if the coloring determined by the vector is a proper, edge 3-coloring. There is no color vector valid for all trees. In particular, we have the following which is only a check of small number of cases.

Lemma 5.3. *There are five finite, binary trees with four leaves and there is no color vector that is valid for all five.*

There are color vectors that are valid for no trees. We say that a color vector is *acceptable* if there is a finite, binary tree for which it is valid. In Section 12 we will prove that a color vector is acceptable if and only if the vector is not a constant vector and does not sum to zero.

A finite, binary tree T with n leaves gives a way to parenthesize a sum of n variables. A leaf corresponds to a single (unparenthesized) variable. If $T = A \wedge B$ where A and B have j and k leaves respectively, then $j + k = n$, the tree A parenthesizes the sum of the first j variables to give an expression E_A , the tree B parenthesizes the sum of the last k variables to give an expression E_B and we define T to correspond to the expression $(E_A + E_B)$. The first sentence of the following is argued inductively and the second follows from the associativity of the addition in $\mathbf{Z}_2 \times \mathbf{Z}_2$.

Lemma 5.4. *If \mathbf{c} is a color vector for finite, binary tree T , then the root color determined by \mathbf{c} is simply the sum of the colors in \mathbf{c} . In particular this color depends only on \mathbf{c} and not on the structure of T .*

We now turn to pairs of trees.

5.4. Coloring binary tree pairs. Recall that given a pair (D, R) of finite trees it is always assumed that the trees have the same number of leaves.

If (D, R) is a pair of finite, binary trees with n leaves each, and \mathbf{c} is a color vector for both D and R , then we say that \mathbf{c} is *valid for* or *is a coloring of* (D, R) if it

is valid for both D and R . If there is a color vector valid for (D, R) , we say that (D, R) has a coloring.

We now can get a statement that is equivalent to the 4CT. This observation is just a variation on statements whose equivalence to the 4CT follows from Whitney's theorem.

Proposition 5.5. *The four color theorem is equivalent to the statement that every pair of finite, binary trees with the same number of leaves has a coloring.*

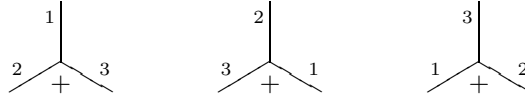
Proof. If (D, R) is the pair of trees associated to $M \in \mathfrak{M}$, then a color vector for the pair determines the same root color for both D and R by Lemma 5.4. Thus the edge colorings of the two trees will match when the map is reconstructed. The other direction is immediate from the construction in Section 2.3. \square

Using arguments from Section 1.1.1, we can introduce the word prime.

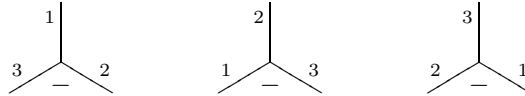
Proposition 5.6. *The four color theorem is equivalent to the statement that every prime pair of finite, binary trees with the same number of leaves has a coloring.*

5.5. Signs. Let T be a finite, binary tree. It will be convenient to view T as drawn in the plane in a way that agrees with the ordering. Assume that T is given a proper, edge 3-coloring from the non-zero elements of $\mathbf{Z}_2 \times \mathbf{Z}_2$. Taking into account the cyclic ordering of the edges around a vertex v of valence three and the fact that one edge (the edge to the parent) is distinguished, there are six possible valid colorings of the edges that meet at v . If the colors 1, 2 and 3 are assigned so that the numbers increase by one as the intervals are visited in the cyclic order, then v is said to have *positive* sign (or color). Otherwise it is said to have *negative* sign (or color).

In case our definition of positive and negative was not clear, we show the positive arrangements



and the negative arrangements.



Thinking of the plus and minus signs as colors gives us a vertex 2-coloring of the vertices of valence 3 in T . To keep the number of references to colorings down, we refer to this kind of coloring as a *sign assignment* on T . It is convenient to have a symbol for a sign assignment, so saying that σ is a sign assignment means σ is a function from the internal vertices of T to the set of two symbols $\{+, -\}$. We will use T^σ to denote a tree with sign assignment σ .

There are no restrictions on a sign assignment. In particular two vertices joined by an edge can be given the same sign, and assigning the same sign to all vertices of valence 3 is an acceptable example of an assignment.

We immediately have the following.

Lemma 5.7. *Given a finite, binary tree T , a sign assignment on T , and an assignment of a non-zero color from $\mathbf{Z}_2 \times \mathbf{Z}_2$ to a single edge of T , then there is a unique, proper, edge 3-coloring for T that agrees with the given information.*

Thus given a proper, edge 3-coloring of a T as above, we get a sign assignment for T , and given a sign assignment for such a T , we get three possible proper, edge 3-colorings that are consistent with the sign assignment. The three different colorings are determined by starting with the three possible different colors for one chosen edge.

5.6. Small examples. The trees shown below are the only finite, binary trees with three leaves.



The following is the result of checking a small number of cases.

Lemma 5.8. *The only color vectors that are simultaneously valid for the pair of trees shown above have the form (a, b, a) where $a \neq b$. In all valid cases, the signs on vertices u and v will be equal, and will be the negatives of the signs on vertices w and x .*

Conversely, if a color vector is valid for the left tree shown above and makes the signs of vertices u and v the same, then the vector is of the form (a, b, a) with $a \neq b$ and the paragraph above applies.

Lastly, the color vector (a, b, a) with $a \neq b$ produces the color b at the root edge of either tree.

5.7. Permuting colors. Two 3-colorings (of anything, a tree, an $M \in \mathfrak{M}$), that differ but can be made the same by permuting the colors of one of the colorings are not different to us in an interesting way. If a finite, binary tree T has a proper, edge 3-coloring and its derived sign assignment, then permuting the colors will either preserve the sign assignment (if the permutation is even) or negate it (if the permutation is odd). Conversely a sign assignment on T together with the negative of the sign assignment will produce all six permutations of an edge 3-coloring of T by running through the three starting colors for a particular edge for both the sign assignment and its negative.

We can try to “normalize” colorings on a finite, binary tree T by insisting that the color of the root edge be 1, and that the sign of the unique child of the root be positive. However, Lemma 5.8 shows that we might need to consider colorings that violate this normalization if we are to look for color vectors that are simultaneously valid for pairs of trees.

Thus if the coloring of a finite, binary tree assigns the color 1 to the root edge, we say it is *positive normal* if the sign of the child of the root is positive and *negative normal* if the sign of the child of the root is negative.

Note that there is a one-to-one correspondence between sign assignments of a finite, binary tree and proper, edge 3-colorings of that tree that use the color 1 for the root edge. The normalization and this one-to-one correspondence will allow us to discuss sign assignments as if they are edge 3-colorings and vice versa.

We apply this to counting colorings of pairs of finite, binary trees. Given a coloring of a pair (D, R) of finite, binary trees, there is a unique way to permute the colors so that the coloring of D is positive normal. Thus the number of colorings of (D, R) modulo the action of S_3 on the colors is the number of colorings in which

the coloring of D is positive normal. When we count colorings of a pair of finite, binary trees, we will always be counting modulo the action of S_3 on the colors. Statements that colorings are unique will be made with this convention.

5.8. The dual view. The material above applied to triangulated polygons is presented in [11]. In the setting of trees, the driving data is an edge coloring, but for triangulated polygons it is easier to start with vertex colorings. Let T be a triangulation of a polygon. Then (T, c) is a coloring of the vertices. This corresponds to the “color vector” of Section 1.2.2.

From (T, c) , we get (T, d) where d is a coloring of the internal edges of T where each edge is colored by the difference of the colors of its endpoints. Given an edge coloring (T, d) and a color of a single vertex, we recover the vertex coloring (T, c) from which (T, d) is derived.

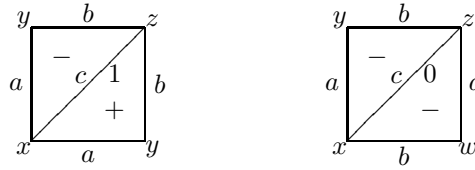
From (T, d) , we get (T, σ) where σ is a sign assignment or signature giving a plus or minus sign to each triangular face depending on the cyclic order of the colors of the edges around the face. Given a signature (T, σ) and a color of a single edge, we recover the edge coloring (T, d) from which (T, σ) is derived. If instead of one edge color, we are told the colors of two adjacent vertices, we recover not only (T, d) but also the vertex coloring (T, c) .

From (T, σ) , we get (T, v) where v is a “valuation” on the edges taking values in $\{0, 1\}$ and an edge is assigned 0 if the signs of the two triangles that share the edge are the same, and 1 otherwise. We build back to (T, σ) , (T, d) and (T, c) given a valuation (T, v) and, respectively, the sign of one triangle, the colors of two edges that share a vertex, or the colors of three vertices of a triangle.

A remark from [11] is worth isolating as a separate statement.

Lemma 5.9. *If T is a triangulation of a polygon and (T, c) and (T, v) are a vertex coloring and its derived edge valuation, then c uses only three colors if and only if all values of v equal 1.*

Proof. This follows inductively from the fact that it holds for squares. The figure below holds all the relevant information. In it $a = x - y$, $b = y - z$, $c = x - z$ are all in $\mathbf{Z}_2 \times \mathbf{Z}_2$ and x, y, z and w are all different. \square

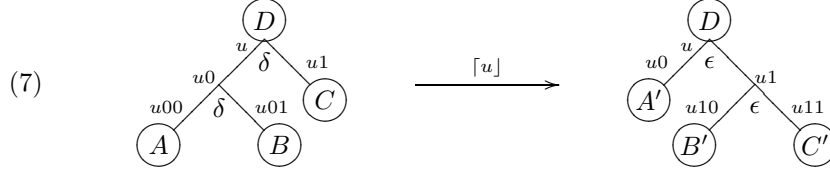


6. COLORED ROTATIONS AND COLORED PATHS

From Proposition 5.5, we know that we want to color pairs of trees. It should be easier to color pairs of trees if the structures of the two trees are closely related. In our setting, the most closely related trees pairs are those that are the endpoints of an edge in an associahedron. Thus we start with such pairs and build from there.

6.1. Colored rotations. We assume a color vector that is valid for both trees shown in (7). Note that these will be two trees that are connected by an edge in some associahedron. The left-right order on the leaves of the two trees matches, each in left-right order, leaves of D to the left of u in the two trees, leaves of A to

those of A' , leaves of B to those of B' , leaves of C to those of C' and leaves of D to the right of u in the two trees. It follows that the three colors at $u00$, $u01$ and $u1$ in the left tree, equal in that order the colors at $u0$, $u10$ and $u11$ in the right tree. From Lemma 5.8, the signs at $u0$ and u , the pivot vertices of $[u]$, in the left tree are equal. Let δ be this common sign. Lemma 5.8 also says that the signs at u and $u1$, the pivot vertices of $[\bar{u}]$, in the right tree are also equal, and if ϵ is this common sign, then $\epsilon = -\delta$. This is illustrated below.



We call the rotation illustrated in (7) a *signed rotation*. This is the admissible transplantation of [13] and is dual to the signed diagonal flip of [8]. The data for such a rotation can be confined to the sign assignment of the two trees and does not need to include either an edge coloring or a color vector. The requirement on the sign assignments is that the signs of the relevant pivot vertices in the two trees are as shown above with $\epsilon = -\delta$, that the isomorphisms from A , B and C to, respectively, A' , B' and C' preserve signs on all non-root vertices, and that the signs in the common subtree D in the two trees are identical. It follows from the remaining provisions of Lemma 5.8 that if there is a signed rotation between two trees, then there is a color vector valid for the two trees.

As before, we use $[u]$ to denote the signed rotation illustrated above. We think of $[u]$ as a function whose value on the left tree in (7) is the right tree.

Extracting more from Lemma 5.8, if there is a tree T with sign assignment as illustrated by the left tree in (7) with signs on u and $u0$ equal, then the rotation $[u]$ can be applied to T to give the tree illustrated by the right tree in (7). In such case, we say that $[u]$ is a *valid rotation* for T . Similarly, we can say that $[\bar{u}]$ is valid for a tree as illustrated by the right tree in (7).

The following is contained in the discussion above.

Lemma 6.1. *Let T and S be vertices in some associahedron A_d connected by an edge. Let $[u]$ be the rotation (unsigned) taking T to S . Let \mathbf{c} be a color vector valid for T and give T the sign assignment derived from \mathbf{c} . Then the following are equivalent.*

- (1) *The rotation $[u]$ is valid for T as a signed rotation.*
- (2) *The color vector \mathbf{c} is also valid for S .*

If T is a finite, binary tree with a sign assignment is given and $[v]$ is a rotation valid for T , then $T[v]$ will denote the result of the rotation.

If w is a word in rotation symbols and if T is a finite, binary tree with a sign assignment, then Tw will be defined inductively as $(Tw')[v]$ if $w = w'[v]$, if Tw' is defined, and if $[v]$ is valid for Tw' . In such case we say that w is a *valid path of signed rotations* for T . We use the word *path* since the movement from tree to tree using the symbols that make up w travels along a path of edges in an associahedron.

We have the following.

Proposition 6.2. *If T is a finite, binary tree with a sign assignment, and w is a path of signed rotations that is valid for T , then there is a color vector that is valid for the pair (T, Tw) .*

We can define the term *valid* to be applied to a path of rotation symbols if it is valid for some finite, binary tree with a sign assignment. We will show in Section 11 that the validity of a path can be detected directly from the path without bringing in a tree with a sign assignment to test it on.

6.2. A converse to Proposition 6.2. A converse to Proposition 6.2 must be stated carefully. A path of signed rotations that is valid for a tree T must start with a sign assignment for T that has at least two adjacent vertices with the same sign. A sign assignment for a tree T that is proper in the sense that every pair of adjacent vertices has opposite sign will be called *rigid* since it allows no signed rotations. Rigidity will be discussed further in Section 9. A sign assignment that is not rigid will be called *flexible*.

The following is a rephrasing of the main result of [11].

Theorem 6.3. *If \mathbf{c} is a color vector valid for the tree pair (D, R) and the sign assignment derived from \mathbf{c} on either D or C is flexible, then there is a path w of signed rotations valid for D so that $Dw = C$.*

6.3. The first signed path conjecture. The following appears in [13] and [8]. It will be restated in Section 11.2 after we verify the claim made above that the validity of a path can be detected directly from the path.

Conjecture 6.4. *For every pair of finite, binary trees (D, R) with the same number of leaves, there is a sign assignment of D and a word w of rotation symbols valid for D so that $Dw = R$.*

It follows from Propositions 6.2 and 5.5 and Theorem 6.3 that Conjecture 6.4 is equivalent to the 4CT.

6.4. Rigidity basics. In a rigid sign assignment on a tree, the sign of the vertex \emptyset determines the entire sign assignment. Thus there are two possible rigid sign assignments for a given tree and we refer to a rigid sign assignment on a tree as its *positive rigid* sign assignment if the vertex \emptyset is positive, and *negative rigid* sign assignment otherwise. A color vector will be called positive rigid for a tree T if it leads to the positive rigid sign assignment on T , and negative rigid for T if it leads to the negative rigid sign assignment on T . A color vector is called flexible for T if it is neither positive rigid, nor negative rigid for T .

The following shows that the words “for T ” are superfluous.

Proposition 6.5 (Trichotomy). *Let \mathbf{c} be an acceptable color vector of length n and let A be the set of all trees with n leaves for which \mathbf{c} is valid. Then exactly one of the following holds.*

- (1) *The vector \mathbf{c} is positive rigid for every T in A .*
- (2) *The vector \mathbf{c} is negative rigid for every T in A .*
- (3) *The vector \mathbf{c} is flexible for every T in A .*

Proof. This is easier to argue from the dual view as laid out in Section 5.8.

A color vector gives an edge coloring to the tree and thus an edge coloring to the dual triangulated polygon P . Note that a rigid coloring (either positive or negative)

corresponds to an edge coloring on P that leads to a derived edge valuation that assigns 1 to all edges.

Given a color c of a single vertex x , this edge coloring of P determines a vertex coloring of P . Changing the given color on x can be viewed as adding an element s of $\mathbf{Z}_2 \times \mathbf{Z}_2$ to c to give the color $c + s$ on x . The coloring determined by the edge coloring and the color $c + s$ on x is obtained by adding s to all the colors obtained from the color c on x and the edge coloring on P . In particular, either all possible vertex colorings use three colors or all possible vertex colorings use 4 colors. By Lemma 5.9, we see that either \mathbf{c} is rigid for every T in A or \mathbf{c} is flexible for every T in A .

Now assume that \mathbf{c} is rigid for T . The sign of the triangle containing the top edge of its dual triangulated polygon P determines whether \mathbf{c} is positive rigid or negative rigid. By the arguments of the previous paragraph, we can fix the color of one vertex on the top edge. The edge coloring determines the color of the other vertex of the top edge. But this determines the color of the third vertex of the triangle containing the top edge since the coloring is rigid and only three colors are used in the vertex coloring. Thus the sign of that triangle does not depend on the location of the third vertex and does not depend on the particular triangulation of P . \square

6.5. Colored associahedra and color graphs. If A_d is the d -dimensional associahedron, then its vertices are trees with $d + 2$ leaves. A color vector \mathbf{c} with $d + 2$ entries induces a coloring on all the vertices of A_d , although for some of the trees the induced coloring from \mathbf{c} may have at least one zero on an edge and be invalid.

We will call the pair (A_d, \mathbf{c}) a colored associahedron. The subset of vertices of A_d consisting of trees for which \mathbf{c} is valid will be called the colored vertices of (A_d, \mathbf{c}) . The set of colored vertices will be empty if the vector is not acceptable, so we restrict this discussion to acceptable color vectors.

The subgraph of the 1-skeleton of A_d spanned by the colored vertices will be called the *color graph* of (A_d, \mathbf{c}) . Since the number of entries in \mathbf{c} specifies the dimension of the associahedron that it colors, it makes sense to refer to the color graph of (A_d, \mathbf{c}) as the color graph of \mathbf{c} . Note that from Lemma 6.1, the edges of the color graph of \mathbf{c} correspond exactly to the rotations that are valid for trees validly colored by \mathbf{c} .

The following is an immediate consequence of Theorem 6.3 and Proposition 6.5.

Theorem 6.6. *Let \mathbf{c} be an acceptable color vector of length $d + 2$. Then its color graph in A_d is either connected or has no edges.*

7. GROUPS

We introduce two groups. One is widely known as Thompson's group F , and the other, while known and related to F , is less well known. Their relevance is that one way to define their elements is as equivalence classes of pairs of binary trees.

7.1. Thompson's group F . Thompson's group F is a finitely presented, infinite group with many interesting properties. It has many closely related faithful representations and we will add to that list here. While coming up with a new representation of F is almost never interesting or deep, it is often useful. We will give what information we need to proceed and will give references where necessary.

A standard reference for the group is [3].

Recall that given a pair (D, R) of finite trees it is always assumed that the trees have the same number of leaves.

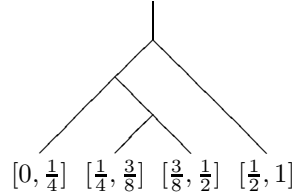
We start with one of the most common representations. The group F can be defined as the group of self homeomorphisms of the unit interval $[0, 1]$ that are defined by pairs (D, R) of finite, binary trees. We will show how to do this.

We associate to each word w in $\{0, 1\}^*$ an interval $i(w)$ in $[0, 1]$. We let $i(\emptyset) = [0, 1]$ and for any w with $i(w)$ defined, we let $i(0w)$ be the image of $i(w)$ under the map $x \mapsto x/2$ and $i(1w)$ be the image of $i(w)$ under the map $x \mapsto (x+1)/2$. It will be convenient to let $m(w)$ be the midpoint of $i(w)$.

Since $i(0)$ is the left half of $i(\emptyset)$ and $i(1)$ is the right half of $i(\emptyset)$, it follows inductively that $i(w0)$ is the left half of $i(w)$ and $i(w1)$ is the right half of $i(w)$.

We apply this by regarding words in $\{0, 1\}^*$ as vertices in \mathcal{T} .

Let T be a finite binary tree. It follows inductively that if L is the set of leaves of T , then $\{i(v) \mid v \in L\}$ is a partition of $[0, 1]$ into subintervals with disjoint interiors so that if v is to the left of w in L , then $i(v)$ is to the left of $i(w)$ in $[0, 1]$. We denote this partition by $P(T)$. Below is a small illustration.



It follows that if v is any vertex in T , then $i(v)$ is the union of all $i(w)$ where w is a leaf of T_v . If this is applied to $v0$ and $v1$, then we have that $m(v)$ is the right endpoint of the union of the $i(w)$ where w is a leaf of T_{v0} and the left endpoint of the union of the $i(w)$ where w is a leaf of T_{v1} .

Let (D, R) be a pair of finite, binary trees having n leaves each and let I_1, \dots, I_n be the intervals in $P(D)$ in left-to-right order in $[0, 1]$, and let J_1, \dots, J_n be the intervals in $P(R)$ in left-to-right order. The homeomorphism $h(D, R)$ defined by (D, R) will take I_i affinely onto J_i , preserving orientation, for each i . The use of D (for domain) and R (for range) is now explained.

Note that $h(D, R)$ is piecewise linear, has only slopes that are integral powers of 2, and that has discontinuities of slope confined to the dyadic rationals $\mathbf{Z}[\frac{1}{2}]$, those rationals of the form $m/2^n$ for integers m and n . The group F is the set of all $h(D, R)$ under composition. It is shown in Lemma 2.2 of [3] that F contains all piecewise linear self homeomorphisms of $[0, 1]$ that have only slopes that are integral powers of 2 and for which the discontinuities of slope are confined to $\mathbf{Z}[\frac{1}{2}]$.

Several pairs of trees can give the same element of F . We can declare two pairs to be h -equivalent if they do. We multiply by composing the homeomorphism and we write the composition from left to right.

We define another relation.

Let A be a finite, binary tree with $n \geq i$ leaves and let u be the address of the i -th leaf in the left-right order. We let $A^{\wedge i}$ denote the finite, binary tree consisting of A and the two extra vertices $u0$ and $u1$.

If (A, B) and (A', B') are two pairs of finite, binary trees then we write $(A, B) \rightarrow (A', B')$ if there is an i so that $A' = A^{\wedge i}$ and $B' = B^{\wedge i}$. We let \sim be the equivalence relation generated by \rightarrow . The equivalence classes are the elements of F . It is shown

in [3] (immediately after the proof of Lemma 2.2 of [3]) that \sim is identical to h -equivalence. The group F is commonly defined as the set of equivalence classes of pairs of finite, binary trees under the relations just defined.

7.2. Carets. It is convenient to call a graph looking like \wedge a *caret*. To make it a formal tree, a root and root edge must be added so that it looks like \bigwedge . We will do so because then binary trees become unions of carets.

In a binary tree containing vertices $w, u, u0$ and $u1$ with w the parent of u , the four element set $\{w, u, u0, u1\}$ forms a caret. We will sometimes refer to u as the *center vertex* of the caret and say that the caret is *centered* at u . In a finite binary tree, the number of carets is the number of internal vertices. Every finite, binary tree is now a union of a finite number of carets if we agree that the trivial tree is a union of zero carets.

We can give words to the construction $A^{\wedge i}$ and say that $A^{\wedge i}$ is obtained from A by “adding a caret to the i -th leaf of A .” We can describe $(A, B) \rightarrow (A', B')$ by saying that we have added a caret to the i -th leaves of both A and B .

7.3. Reduced pairs. A pair (D, R) of finite, binary trees representing an element of F is said to be *reduced* if there is no pair (A, B) with $(A, B) \rightarrow (D, R)$. Should such a pair (A, B) exist, then passing from (D, R) to (A, B) is called a *reduction*.

Reductions are findable. In a finite, binary tree, let us call a caret whose two leaves are both leaves of the tree an *exposed caret*. If $(A, B) \rightarrow (D, R)$, then there will be an exposed caret in D whose leaves occupy the same positions in the left-right order of the leaves of D as the leaves of an exposed caret in R . Conversely, if such “matching” exposed carets exist in a pair of trees (D, R) , then there is a pair (A, B) such that $(A, B) \rightarrow (D, R)$.

Each element of F is represented by a unique reduced pair (shown following Lemma 2.2 in [3]) and any pair will be reduced to a reduced pair in the same equivalence class by any random sequence of reductions taken as far as possible.

7.4. The multiplication. Since we compose the homeomorphisms of $[0, 1]$ that form the elements of F from left to right, it is clear that if (A, B) and (B, C) are two pairs of binary trees, then $(A, B)(B, C) = (A, C)$ as elements of F . We need to deal with the more general situation $(A, B)(C, D)$. Using the equivalence relation \sim derived in Section 7.1 from the relation \rightarrow , we replace $(A, B)(C, D)$ by $(A', B')(C', D')$ where $(A, B) \sim (A', B')$, where $(C, D) \sim (C', D')$, and where $B' = C'$. Now of course the product is (A', D') .

The required equality $B' = C'$ can be achieved because the relation \rightarrow allows us to add a caret at any leaf we please and then, if desired, add more. In particular, we can get $B' = C' = B \cup C$. (See the remarks at the end of Section 3.5.)

The multiplication is algorithmic (and the word problem is solvable) in the sense that if f, g and h are given, then it is possible to tell if $fg = h$ is true. One computes a pair for fg as just described, and then reduces the result and reduces any pair for h using the comments in Section 7.3. The equality $fg = h$ holds if and only if the resulting reduced pairs are identical.

An illustration is essential here. Below we show the calculation of the square of the smallest tree pair that represents the element $[\emptyset]$. We omit the root $*$ and the

root edge to simplify the figures.

$$\begin{aligned}
 (8) \quad & \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} , \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} , \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) \\
 &= \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} , \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} , \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) \\
 &= \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} , \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right)
 \end{aligned}$$

The first pair of trees on the first line is related by \rightarrow to the first pair of trees on the second line. The second pairs of trees on the two lines are similarly related. See the end of Section 7.1.

Regarding trees as unions of carets lets us give more detail to this calculation. We pause to discuss set operations on finite binary trees before returning to the example.

7.5. Unions, intersections and differences of trees. We elaborate on remarks made at the end of Section 3.5.

If B and C are finite, binary trees in \mathcal{T} , then it is easier to work with expressions such as $B - C$ and $C - B$ if we think of B and C as unions of carets. The unions of carets point of view makes no change when discussing $B \cup C$ and $B \cap C$, so there is no disadvantage to this view.

We say that two carets are *adjacent* if they share an edge. If two carets are adjacent, there will be one shared edge and it will be a root edge of one caret and a leaf edge of the other. This allows us to talk about a component of a set of carets as the union of a set of carets whose adjacency graph is maximally connected.

If we define $C - B$ to be the set of carets in C that are not in B , then $C - B$ breaks into components with one component for each leaf of $B \cap C$ that is not a leaf of C . If v is such a leaf, it is a leaf of B and the corresponding component of $C - B$ will be the subtree C_v . The subtree C_v will share its root edge with the leaf edge of B (and of $B \cap C$) that impinges on v .

Similar remarks apply to components of $B - C$ and leaves of $B \cap C$ that are not leaves of B .

We now have that $B \cup C$ breaks into three parts, the tree $B \cap C$, a finite collection of trees making up $C - B$ with one tree for each leaf of $B \cap C$ that is not a leaf of C , and a finite collection of trees making up $B - C$ with one tree for each leaf of $B \cap C$ that is not a leaf of B . Any leaves of $B \cap C$ not yet accounted for are leaves of both B and C .

We can apply this to the calculation of the product $(A, B)(C, D)$. This is equal to (A', D') which we get from $(A', B \cup C)(B \cup C, D')$ chosen so that $(A', B \cup C) \sim (A, B)$ and $(B \cup C, D') \sim (C, D)$. Now the components of $A' - A$ form a finite collection of trees that are in one-to-one correspondence with the components of $C - B$. Let v be a leaf of B that is not a leaf of C with component C_v attached to $B \cap C$ at v and assume v is the i -th leaf in the left-right order of the leaves of B . Let v' be the i -th leaf in the left-right order among the leaves of A . Then $A'_{v'}$ is the component of $A' - A$ attached to A at v' , is the component of $A' - A$ corresponding to C_v , and is isomorphic to C_v . Similar remarks apply to components of $D' - D$.

In the calculation illustrated in (8), let A, B, C, D be the trees in the first line, reading left to right. In the move to the second line, the lone caret in $C - B$ was

attached to the leftmost leaf of A and of B , and the lone caret of $B-C$ was attached to the rightmost leaf of C and of D .

7.6. Another representation. Let $\mathbf{D} = \mathbf{Z}[\frac{1}{2}] \cap (0, 1)$, the set of dyadic rationals between 0 and 1. Each $x \in \mathbf{D}$ has a last “1” in its base 2 representation. The word w in $\{0, 1\}^*$ in the base 2 representation of x from the binary point up to but not including the last 1 has $m(w) = x$. Thus $m : \{0, 1\}^* \rightarrow \mathbf{D}$ is a bijection. If we regard $\{0, 1\}^*$ as the non-root leaves of \mathcal{T} , if \mathcal{T} is given the infix order, and if \mathbf{D} is given the usual order in $[0, 1]$, then m preserves order.

Since \mathbf{D} is dense in $[0, 1]$ and an element $h(D, R)$ in F preserves \mathbf{D} , the element $h(D, R)$ is determined by its action on \mathbf{D} and thus is also determined by the corresponding action on the non-root vertices of \mathcal{T} . We can extend the action to be the identity on the root of \mathcal{T} so that we can stop saying “non-root vertex.” Note that these actions are not tree homomorphisms. We denote the action given by a pair (D, R) by $t(D, R)$.

The “other” representation of F is as the set of permutations $t(D, R)$ on the vertices of \mathcal{T} as (D, R) runs over all pairs of finite, binary trees. This will be our working version of F .

Since this is the version that we work with and since it is traditional to write elements of F as pairs of binary trees, we will drop the t and write (D, R) instead of $t(D, R)$. This starts immediately.

7.7. Actions on finite trees. We write the action of (D, R) to the right of its arguments.

Noting that both D and R are rooted subtrees of \mathcal{T} , we see that the statement of the next lemma makes sense. We will also argue that it is true.

Lemma 7.1. *Let (D, R) be a pair of finite, binary trees with the same number of leaves. Then $(D)(D, R) = R$. Further if the vertices in both D and R are given the infix order, then the restriction of (D, R) to D preserves this order. Lastly, (D, R) takes leaves to leaves and internal vertices to internal vertices.*

Proof. Since $h(D, R)$ takes the j -th interval in $P(D)$ affinely onto the j -th interval in $P(R)$, we have that (D, R) takes the j leaf of D to the j -th leaf of R . Induction using $\hat{}$ shows that, in the infix order of any finite, binary tree, between any two consecutive leaves there is a single internal vertex. This immediately gives the last sentence in the statement. Further, let v_j and v_{j+1} be two consecutive leaves of D in the infix order and let w be the single internal vertex between them. From the information about the functions i and m , we have that $m(w)$ is the right endpoint of $i(v_j)$ and the left endpoint of $i(v_{j+1})$. Similar statements apply to the leaves of S that are the images of v_j and v_{j+1} , and so the image of w is the single internal vertex of R between the images of v_j and v_{j+1} . This completes all claims. \square

Corollary 7.1.1. *For each $d \geq 1$, the vertices of A_d form the objects of a category in which for each pair (D, R) of vertices, the only morphism is (D, R) .*

Proof. Since each (D, R) preserves a linear order, all triangles to commute. \square

7.8. Rotation as action. We apply the above to rotations.

Corollary 7.1.2. *If the rotation $[u]$ takes T to S as shown in (4), then (T, S) takes $u0$ and u in T , respectively, to u and $u1$ in S . That is, $[u]$ takes the pivot vertices of $[u]$ to the pivot vertices of $[\bar{u}]$ so as to preserve the infix order.*

Proof. The positions of $u0$ and u in the infix order on T are the positions, respectively, of u and $u1$ in the infix order on S . \square

Because of Corollary 7.1.2, we identify the rotation $\lceil u \rceil$ from T to S with the permutation (T, S) . In the next proposition, we make use of the notation T^σ to indicate a tree with sign assignment σ .

Proposition 7.2. *Let $\lceil u \rceil$ be the positive signed rotation pictured in (7), let T^σ denote the left tree with its sign assignment, and let S^ρ denote the right tree with its sign assignment. Then for all vertices internal vertices v of T , the rotation $\lceil u \rceil$ preserves all signs except those of the pivot vertices of $\lceil u \rceil$ which it negates. Specifically*

$$\rho(v\lceil u \rceil) = \begin{cases} \sigma(v), & v \notin \{u, u0\}, \\ -\sigma(v), & v \in \{u, u0\}. \end{cases}$$

Similarly, the rotation $\lceil \bar{u} \rceil$ preserves the signs of all internal vertices of S except for negating the signs of the pivot vertices of $\lceil \bar{u} \rceil$.

We define the action of $\lceil u \rceil$ on T^σ to have the action of $\lceil u \rceil$ on the vertices of T as given by Corollary 7.1.2 and the action of $\lceil u \rceil$ taking σ to ρ as given by Proposition 7.2. We can thus say $T^\sigma \lceil u \rceil = T^\rho$.

Note that if T is a tree with sign assignment σ and the pivot vertices of $\lceil u \rceil$ are internal vertices of T , then $T^\sigma \lceil u \rceil$ can be defined using the specifications given in Proposition 7.2 even if σ does not give the same values to the pivot vertices of $\lceil u \rceil$. That is, the action can be defined even if $\lceil u \rceil$ is not valid for T^σ .

7.9. Edge paths in the associahedra. If a pair (D, R) of finite, binary trees each of which has $d + 2$ leaves represents an element of F , then it also gives a pair of vertices in the associahedron A_d . Since each edge of A_d is a rotation, an edge path in the 1-skeleton of A_d gives a finite string of rotations. If $\lceil u_1 \rceil \lceil u_2 \rceil \cdots \lceil u_n \rceil$ is such a path from D to R , then it follows from Lemma 7.1 and the convention given after Corollary 7.1.2 that the product $\lceil u_1 \rceil \lceil u_2 \rceil \cdots \lceil u_n \rceil$ in F equals (D, R) as an element of F .

7.10. Presentations. From the statement that the 1-skeleton of each associahedron is connected, we get that the rotations generate F . We can discuss relations.

We need some notation. Let u and v be elements of $\{0, 1\}^*$. If u is a prefix of v , we write $u|v$. If neither of u nor v is a prefix of the other, we write $u \perp v$.

We can now write specifically the action of the rotation $\lceil u \rceil$ on $\{0, 1\}^*$. We use w to represent an arbitrary element of $\{0, 1\}^*$.

$$v\lceil u \rceil = \begin{cases} v, & u \perp v, \\ v, & v|u, \\ u0w, & v = u00w, \\ u10w, & v = u01w, \\ u11w, & v = u1w, \\ u, & v = u0, \\ u1, & v = u. \end{cases}$$

We leave it to the reader to verify certain relations. The more straightforward of the relations are conjugacy relations. Because of the first two lines in the definition

of $v[u]$, the “support” of $[u]$ is all v where $u|v$. Thus a nice conjugacy relation exists between $[u]$ and $[v]$ if the support of $[v]$ is treated nicely by $[u]$. This leads to the following which hold for all u and v as specified and are easy to verify. We use a^b for $b^{-1}ab$.

(F1) $[v]^{[u]} = [v]$ if $v \perp u$.

(F2) $[v]^{[u]} = [v[u]]$ if $u|v$ and v is neither $u0$ nor u .

There are relations between $[u]$ and $[u0]$. The last set should be compared to the copy of A_2 shown in (5) in Section 4.3.

(F3) $[u0][u][u1] = [u][u]$.

It is shown in [7] that F is presented with the set of all $[u]$ for $u \in \{0,1\}^*$ as generating set and with all relations in (F1), (F2) and (F3) as the relation set. The generators in $\{[u] \mid u \in \{0,1\}^*\}$ are called the *symmetric generators* in [7] and we shall do the same.

The argument in [7] assumes little about F . A different proof that uses more information about F is to argue that the relations above imply that F is generated by the set $[1^*]$. In fact, one shows that F is generated by $[\emptyset]$ and $[1]$. The relations in (F2) contain all relations needed to present F using either $[1^*]$ as generators or $[\emptyset]$ and $[1]$. See [3]. (The rotation $[1^i]$ is the inverse of the generator denoted X_i in Figure 8 of [3].) This makes our representation of F a homomorphic image of the group presented in [3]. It is a fact that F has no proper non-abelian quotients (Theorem 4.3 of [3]). That is, any non-trivial normal subgroup of F contains the commutator subgroup of F . Since our representation is non-abelian, the homomorphism from the group in [3] to our representation is an isomorphism.

7.11. Sizes of trees in tree pairs. The notion of size is slippery in F . Since an element of F is an equivalence class of pairs of trees, an obvious candidate is some measure of tree pairs that is minimized over all pairs that represent an element. A typical measure is the number of carets (which equals the number of internal vertices) used in each tree of the tree pair. This will be minimized by the unique reduced pair representing an element.

This measure is only moderately well behaved. In general it is complicated to predict the size of the result of a multiplication without actually doing the multiplication. In particular it is complicated to predict the size of the result of multiplying an element by a rotation (one of the generators used above) without actually doing the multiplication.

We deal with this by partly ignoring it. We will not be concerned with minimal representatives of elements of F and only concerned with the size of the representative that we have in hand. A single pair (D, R) is a pair of vertices on a specific associahedron A_d . Edges in the associahedron A_d are rotations that apply to trees of the size of D and R . Only finitely many of the generators in $\{[u] \mid u \in \{0,1\}^*\}$ are available among the edges of A_d . If a generator outside this finite set is to be used, then a different (larger) representative will need to be chosen (a move to a larger associahedron takes place) and this will be duly noted. These considerations will appear below in Section 8.5.

7.12. The group E . We now describe the other group we consider.

The relations in (F1) and (F2) involve four generators each and will be referred to as the “square relations,” while the relations in (F3) involve five generators and will be referred to as the “pentagonal relations.” We will see that the pentagonal

relations behave very differently from the square relations when the generators are interpreted as signed rotations.

We define the group E as the group presented by the same generating set as F and with the relations given by (F1) and (F2). The group E is finitely presented, but that fact is neither surprising nor important here. The group E is an extension of F . We use the letter E since it is the first letter of “extension,” since it is next to F in the alphabet, and since the other neighbor of F in the alphabet has already been used for groups closely related to F .

The group E has been mentioned in conversations by several people familiar with F but there has never been any compelling reason to consider it. Part of the charm of Thompson’s groups or its close relatives is that they are either simple or close to it. Groups that are farther from simple are often not considered.

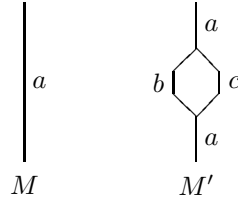
We will bring the group E into the discussion when we investigate valid paths of signed rotations in Section 11.

8. COLORS AND THE GROUP OPERATIONS

We extract some information about colorings from the multiplicative structure of F . We ultimately discover that most of the difficulty boils down to problems that live in the associahedra.

8.1. Coloring representatives of a single element of F . Representatives of elements of F are pairs of finite, binary trees. We say that such a pair has a coloring if there is a color vector valid for the pair. From Proposition 5.5, the 4CT implies that every representative of every element of F has a coloring. For now, we do not assume the 4CT.

If $(A, B) \rightarrow (A', B')$, then the maps M and M' that, respectively, correspond to the pairs differ in one respect shown below. The figure only shows where they differ.



The letters show the only possible arrangements of a proper, edge 3-coloring in the portion shown. It follows immediately that M has a proper, edge 3-coloring if and only if M' has a proper, edge 3-coloring, independent of the 4CT. Thus (A, B) has a coloring if and only if (A', B') has a coloring, independent of the 4CT.

From this it follows that (independent of the 4CT) there is a well defined notion of an element of F having a coloring. We have the next variant of Proposition 5.5.

Proposition 8.1. *The four color theorem is equivalent to the statement that every element of F has a coloring.*

8.2. Extending the set of colored elements. In Corollary 2.6 of [3] it is shown that $[\emptyset]$ and $[1]$ generate F . We know that the set of elements of F with colorings is non-empty. The identity certainly has a coloring. If a non-identity element is sought, then Lemma 5.8 says that $[\emptyset]$ has a coloring. It is trivial to show that $[1]$ has a coloring. Since every element of F is a product of copies of $[\emptyset]$, $[1]$ and

their inverses, and a coloring of an element is also a coloring of the inverse of that element, we have the following.

Theorem 8.2. *The four color theorem is equivalent to the statement that the set of elements of F with colorings is closed under right multiplication by $[\emptyset]$, $[\bar{\emptyset}]$, $[1]$ and $[\bar{1}]$.*

We will see later that this theorem has limited effect.

We note that the set of elements of F with colorings is clearly closed under inversion (independent of the 4CT). The dihedral group of order $2(d+3)$ acts on A_d and thus on pairs of its vertices (elements of F). The set of elements of F with colorings is also closed under this action (independent of the 4CT). We have not explored how much this extends the set of colored elements of F , but there are reasons for not expecting much from such extension. See the remarks after (9) below.

8.3. The compatibility lemma. Consider a product $(A, B)(C, D)$ of elements of F . Assume that \mathbf{a} is a color vector valid for (A, B) and that \mathbf{c} is a color vector valid for (C, D) . The vector \mathbf{a} determines a proper, edge 3-coloring of B , and \mathbf{c} determines a proper, edge 3-coloring of C . We say that \mathbf{a} and \mathbf{c} are compatible on $B \cap C$ if the colorings they determine on B and C , respectively, agree on $B \cap C$.

Lemma 8.3 (Compatibility). *If the vector \mathbf{a} is valid for the pair (A, B) , the vector \mathbf{c} is valid for (C, D) , and the vectors are compatible on $B \cap C$, then there is a coloring of $(A, B)(C, D)$.*

Proof. We will refer to color vectors as indexed by leaves rather than indexed by integers. This suffices to specify the vector.

We let the vector \mathbf{d} indexed over the leaves of $B \cup C$ have the color of \mathbf{a} on a leaf of $B \cup C$ that is a leaf of B but not a leaf of C and have the color of \mathbf{c} on the remaining leaves. Note that the second choice agrees with the color from \mathbf{a} on the leaves of $B \cup C$ that are leaves of both B and C since such leaves are also leaves of $B \cap C$.

The discussion that follows is based on the discussion in Section 7.5. We view the product $(A, B)(C, D)$ as computed by $(A', B \cup C)(B \cup C, D')$ as describe in that section.

For a component C_v of $C - B$ that is attached to $B \cap C$ at leaf v of B , the colors of \mathbf{d} on those leaves are those of \mathbf{c} . Thus the root color of C_v from \mathbf{d} is the root color of C_v from \mathbf{c} . But the root edge of C_v is the leaf edge of $B \cap C$ impinging on v . By assumption, this edge gets the same color from \mathbf{a} and \mathbf{c} . This common color is also the color that \mathbf{a} assigns to the leaf edge in A for the leaf v' that has the same left-right position in A as v has in B . But A' has an isomorphic copy of C_v attached to A at v' and the color derived from \mathbf{d} at the leaf edge to v' will agree with the color assigned there by \mathbf{a} . It follows that the edge coloring on A' derived from \mathbf{d} will agree with the color on A derived from \mathbf{a} and agree with the colors on components of $A' - A$ derived from \mathbf{c} . Thus the color vector \mathbf{d} will be valid for A . Similar remarks apply to D' . \square

8.4. Vines. The trees in this section are all finite, and binary.

A vine is a tree with exactly one exposed caret. Note that the adjacency graph of the carets in a vine will have no branches.

If T is a tree and V is a vine, then $T \cap V$ and $V - T$ are vines. In particular, $V - T$ is connected.

The right vine is a vine where all the internal vertices are of the form 1^* . The reader can define the left vine, but we will rarely refer to it.

If v is a vertex in \mathcal{T} , then there is a unique vine V_v in \mathcal{T} which has v as an internal vertex and for which the children of v are the exposed leaves of V_v . Note that with this notation, we have that the rotation $\lceil u \rceil$ is given by the pair (V_{u0}, V_{u1}) . Note also that $V_{u0} \cap V_{u1} = V_u$.

8.5. Action of rotations on colored pairs. In the next discussion a rotation such as $\lceil u \rceil$ will be viewed as both an element of F (and thus represented as an ordered pair of trees) and an action on the vertices of \mathcal{T} and some of its finite subtrees.

Let a pair (D, R) of finite, binary trees represent an element of F and assume that \mathbf{c} is a valid color vector for the pair. We are interested in the influence of \mathbf{c} on $(D, R)\lceil u \rceil$ for some rotation $\lceil u \rceil = (V_{u0}, V_{u1})$. The discussion of $\lceil \bar{u} \rceil$ is similar.

There are several cases depending on the relationship between V_{u0} and R .

If V_{u0} is a subtree of R , then it is immediate that $(D, R)\lceil u \rceil = (D, R\lceil u \rceil)$ in which both views of $\lceil u \rceil$ are used. From Section 6.1 we know that \mathbf{c} is valid for the pair $(R, R\lceil u \rceil)$ if and only if the signs of u and $u0$ agree in R under \mathbf{c} . Thus \mathbf{c} is valid for $(D, R)\lceil u \rceil = (D, R)(V_{u0}, V_{u1})$ if and only if the signs of u and $u0$ agree in R under \mathbf{c} .

We turn to the case where V_{u0} is not a subtree of R .

If V_{u0} is not a subtree of R , then the behavior depends on the number of carets of $V_{u0} - R$. We consider first the case in which $V_{u0} - R$ consists of only one caret.

If $V_{u0} - R$ consists of one caret, then this caret is centered at $u0$, and $u0$ is a leaf of R . If $u0$ is the i -th leaf of R in left-right order, then it is immediate that $(D, R)\lceil u \rceil = (D^{\wedge i}, (R^{\wedge i})\lceil u \rceil)$ and we become interested in the pair $(R^{\wedge i}, (R^{\wedge i})\lceil u \rceil)$ shown below in which $\epsilon = -\delta$.

$$(R^{\wedge i}, (R^{\wedge i})\lceil u \rceil) = \left(\begin{array}{c} \text{Diagram 1} \end{array}, \begin{array}{c} \text{Diagram 2} \end{array} \right)$$

Extending the color of \mathbf{c} to $R^{\wedge i}$ can be done in either of two ways. However, if the extension is to be valid for $(R^{\wedge i}, (R^{\wedge i})\lceil u \rceil)$, then the signs at u and $u0$ must agree. Since u is internal to R , its sign is fixed. Thus only one of the two ways to extend \mathbf{c} to $R^{\wedge i}$ is valid for the pair. This extension will be valid for $(D, R)\lceil u \rceil$. We thus have a process that gives a coloring to $(D, R)\lceil u \rceil$ from a coloring of (D, R) under the assumption that $V_{u0} - R$ consists of one caret.

Still under the assumption that $V_{u0} - R$ consists of one caret, let us assume that $(D, R)\lceil u \rceil = (D^{\wedge i}, (R^{\wedge i})\lceil u \rceil)$ has a coloring. Since the caret in $D^{\wedge i}$ attached at the i -th leaf of D is exposed, its leaf edges must be assigned different colors. It follows that the leaf edges to $u0$ and $u10$ in $(R^{\wedge i})\lceil u \rceil$ are assigned different colors, and from that it follows that the signs at u and $u1$ in $(R^{\wedge i})\lceil u \rceil$ are equal. By considerations similar to the paragraph above, we can build a coloring valid for (D, R) .

The processes of the previous two paragraphs are inverses to each other and under the assumption that $V_{u0} - R$ consists of one caret, we have a one-to-one correspondence between the colorings valid for (D, R) and the colorings valid for $(D, R)[u]$. This recovers Proposition 11 of [5].

We leave it to the reader to show that if $V_{u0} - R$ consists of more than one caret, then a coloring for (D, R) can be converted, non-uniquely, to a coloring of $(D, R)[u]$.

We note that if $V_{u0} - R$ is empty, then the number of carets involved in (D, R) is the same as the number of carets involved in $(D, R)[u] = (D, R[u])$. If $V_{u0} - R$ is not empty, then the number of carets in $(D, R)[u]$ is greater than the number of carets in (D, R) . Note that we are ignoring the fact that different pairs representing the same element of F may use different numbers of carets. Here we are referring to pairs and not elements of F .

We say that multiplication $(D, R)[u]$ is *increasing* if $V_{u0} - R$ is not empty, and *minimally increasing* if $V_{i0} - R$ consists of exactly one caret. We say a chain $[u_1] \cdots [u_n]$ of rotations is increasing if letting $p_i = [u_1] \cdots [u_i]$ makes $p_i[u_{i+1}]$ increasing for $1 \leq i < n$. We say the chain is minimally increasing if each multiplication is minimally increasing.

In the above, similar analysis and terminology applies to multiplication by $[\bar{u}]$ and also rotations (by either $[u]$ or $[\bar{u}]$) applied on the left. (Earlier we said actions are written on the right, but $[u]$ and $[\bar{u}]$ are also elements of F and can be multiplied on the left of another element of F .)

From Lemma 5.8, we know that $[\emptyset]$ and $[\bar{\emptyset}]$ each have a unique coloring. All other rotations have colorings which are not unique. Recall from Section 5.7 that we count colorings modulo permutations of the colors. From this and the discussions above, we get the following.

Theorem 8.4. *If an element of F is represented as a minimally increasing chain starting with $[\emptyset]$, then it has a unique coloring. If an element of F is represented as an increasing chain, then it has a coloring.*

The limitations of this result can be seen by noting that the result of an increasing multiplication by a rotation must have an exposed caret of one tree have its leaves match to leaves of adjacent carets (carets that share an edge) in the other tree. The following shows that not every tree pair has this configuration.

$$(9) \quad \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} , \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right)$$

The trees in (9) are those in (3). It follows that the action of the dihedral group of order 12 by root shifts and reflections can only take the element in (9) to itself or its inverse. In particular the element in (9) is the image of no element under the action other than itself or its inverse.

8.6. Application to specific subsets of F . The set of elements of F of the form (D, V) where V is a right vine forms a monoid of F that is often called the positive monoid of F . This is because any element (D, R) of F is of the form $(D, V)(V, R)$ and is thus of the form pn^{-1} where p and n are from the positive monoid. Elements of the positive monoid are referred to as positive elements of F .

With prime as defined at the end of Section 2.3, we have that the prime, positive elements of F are of the form (D, V) where V is a right vine and the right subtree of D is trivial.

Theorem 8.5. *The positive elements of F have colorings and the prime positive elements of F have unique colorings.*

Proof. We leave it as an easy exercise to argue that positive elements of F are represented as increasing chains and that the prime positive elements are represented as minimally increasing chains. \square

Theorem 8.5 has a very straightforward proof using the dual (triangulations) view. See Section 2.4. We leave this an exercise to the reader.

Computer calculations indicate that the fraction of the uniquely colored elements of F represented by positive primes goes to zero as the size of the trees in the pair increases.

We note that the cardinality portions of the conclusions of Theorems 4, 6 and Proposition 11 of [5] follow from Theorem 8.5. The cardinality parts of the conclusions of Theorems 5, 7, 8, 10 and 12 of [5] follow from Theorem 8.5 and (1) in Section 1.1.1.

8.7. Further considerations. From Theorem 8.4 we see that the difficulties in attempting a proof of the 4CT by coloring pairs of trees are found in deducing the existence of a coloring of a tree pair from a coloring of a pair of the same size. Proposition 6.2 is our approach to this problem. For a given tree pair, Proposition 6.2 and its companion, Conjecture 6.4, live on the associahedron that has the trees of the pair as vertices.

We thus concentrate on edge paths in an associahedron. Rigid colorings are beyond the edge path approach. These were discussed in Section 6.4 and will be discussed more in Section 9. Then Section 10 investigates edge paths under the assumption that the colorings are not rigid. The study of edge paths is completed in Section 11 which gives the argument, promised after the statement of Proposition 6.2, that the “validity” of an edge path can be detected without bringing in a sign assignment of a tree to test it on.

9. RIGID COLORINGS

9.1. Depth condition. Let w be a vertex in \mathcal{T} other than the root $*$ regarded as a word in the alphabet $\{0, 1\}$. We use $|w|$ to denote the length of w as a word. We refer to $|w|$ as the level or depth of w . If (D, R) is a pair of binary trees of n leaves each, with (d_1, d_2, \dots, d_n) and (r_1, r_2, \dots, r_n) the leaves, respectively, of D and R in left-right order, then we say that the pair satisfies the *parity condition* if for $1 \leq i \leq n$, the parity of $|d_i|$ and $|r_i|$ agree.

Proposition 9.1. *Let (D, R) be a pair of finite, binary trees. Then (D, R) has a rigid coloring if and only if (D, R) satisfies the parity condition.*

Proof. From Lemma 5.9, we know that (D, R) has a rigid coloring if and only if the map created from $D \cup R$ as described in Section 2.3 has a proper, face 3-coloring. From Theorem 2-5 of [19], we know that every face of the map has an even number of edges. If we apply this to the two faces containing the root edges of D and R , we see that the parity condition holds for $i = 1$ and $i = n$. The rest follow inductively from $i = 1$ by consideration of the other faces. \square

9.2. The positive rigid pattern. Let (D, R) be a pair of trees with a rigid coloring. Then by Proposition 6.5 they possess a common positive rigid coloring and a common negative rigid coloring. One is just the negative of the other and it is not worth discussing both. So we concentrate on the positive rigid coloring.

Proposition 9.2. *The set of elements of F that have a rigid coloring forms a subgroup of F .*

We will use F_4 to denote the subgroup of F consisting of those elements with rigid colorings. The reason for the notation comes from the next proposition. The notation is reasonably common.

Proposition 9.3. *The group F_4 consists of those elements of F which when viewed as self homeomorphisms of $[0, 1]$ use only slopes that are integral powers of 4.*

Proof. This is an immediate consequence of Proposition 9.1. \square

9.4. Characterizing positive rigid color vectors. We can now give a characterization of the positive rigid color vectors that we will use in Section 15.5 when we count rigid color vectors.

Proposition 9.4. *The positive rigid color vectors (modulo the action of S_3 on the colors) are the non-constant vectors \mathbf{c} that sum to 1 and for which no prefix sums to 3.*

Proof. With the convention that 1 is the root edge color, we have that the sum of any vector that we discuss is 1. The assumption that the top internal vertex is positive takes care of the rest of the action by S_3 .

For a pair (D, R) with a common positive, rigid coloring, we consider the proper, face 3-coloring of the map $D \cup R$ compatible with the edge coloring. Each edge has the non-zero color that is not the color of either face impinging on the edge. It follows that all the edges that impinge on vertices in the boundary of a face F but are not edges in that boundary are given the color of F .

The face that impinges on the root edges of $D \cup R$ to the left of the trees (as drawn in the plane) must have the color of the right descending edge from the top internal vertex. In the positive rigid tree, this color is 3. The remaining faces use colors 1, 2, and 3. All faces touch the leaves of the trees and the leaf colors give the differences between the colors of two consecutive faces. If any prefix sums to 3, then with the left face being colored 3, there will be a face colored zero and conversely. \square

10. COLOR GRAPHS, ZERO SETS, SHADOW PATTERNS, AND LONG PATHS

10.1. Zero sets. We return to color graphs but from a complementary view. The set of vertices in a colored associahedron (A_d, \mathbf{c}) for which \mathbf{c} is not valid will be called the *zero set* of (A_d, \mathbf{c}) . The name is chosen since a vertex (tree) in the zero set will have an edge colored zero by \mathbf{c} . As with the color graph it makes sense to refer to the zero set of (A_d, \mathbf{c}) as the zero set of \mathbf{c} . Theorem 6.3 implies the following.

Proposition 10.1. *For each $d > 1$, no zero set of any (A_d, \mathbf{c}) with \mathbf{c} acceptable and flexible separates the 1-skeleton of A_d .*

We now discuss zero sets of acceptable color vectors.

Let \mathbf{c} be an acceptable color vector with $d + 2$ entries, and let T be a vertex in A_d . If \mathbf{c} is not valid for T , then let e be an edge that evaluates to zero when \mathbf{c} is applied to the leaves of T . If v is the lower endpoint of e , then T_v has edge e as its root edge and the leaves of T_v form an interval in the totally ordered set consisting of the leaves of T . That is, for some interval $[i, j]$ in $[1, d + 2]$, the leaves of T_v are the i -th through the j -th in the left-right order. The color zero assigned to e is a result of the fact that the sum of the colors in \mathbf{c} from the i -th through the j -th is zero.

This leads to the next definition. If T is a finite, binary tree with $d + 2$ leaves, then for each internal vertex v of T , the leaves of T_v form the *shadow* of v , and the

The set of shadow intervals of all internal vertices of T except for the vertex \emptyset forms the *shadow pattern* of T . The shadow interval of \emptyset includes all of $[1, d+2]$ and gives no information. It is an exercise that T is determined by its shadow pattern. We will see examples shortly.




Given any interval J of length k in $[1, d + 2]$, the set of trees with J in their shadow patterns is a codimension-1 face of A_d of the form $A_{k-2} \times A_{d-k+1}$. From this it follows that the zero set of a given color vector is a union of codimension-1 faces. Not every union of codimension-1 faces is a zero set. In particular not every collection of intervals is some $Z_{\mathbf{c}}$ for some color vector \mathbf{c} . If $[a, b]$ is in $Z_{\mathbf{c}}$, then $[a, b + 1]$ cannot be since then the color of leaf $b + 1$ would have to be zero. Further, if $[a, b]$ and $[b + 1, d]$ are in $Z_{\mathbf{c}}$, then so must $[a, d]$ be in $Z_{\mathbf{c}}$. Lastly, we show a union of codimension-1 faces that separates, in contrast to Proposition 10.1.




Since a codimension-1 face can be specified by an interval of leaf numbers, we will give our union of codimension-1 faces as a set of intervals in $[1, 6]$. The intervals are

These cannot correspond to a zero set of an acceptable color vector because of the presence of intervals $[3, 6]$ and $[4, 6]$.

$$(10) \quad \begin{array}{cccccc} & & & \overline{\hspace{1.5cm}} & & \\ & & & \overline{\hspace{1cm}} & & \\ & & \overline{\hspace{1cm}} & & \overline{\hspace{1cm}} & \\ & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

(11)

	1	<u>2</u>	3	<u>4</u>	<u>5</u>	6
	1	<u>2</u>	3	<u>4</u>	<u>5</u>	6
	1	2	<u>3</u>	<u>4</u>	<u>5</u>	6

	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>	<u>6</u>
	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>	<u>6</u>
	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>	<u>6</u>

Recall that the shadow interval $[1, 6]$ of the vertex \emptyset is not part of any pattern and is not shown.

What must be done now is to verify that any rotation of any of the six trees above either results in another one of the six or a tree in the one of the faces specified by (10). To do this, we look at the effect of rotation on the shadow intervals. Rotations involving small parts of the tree have few possible arrangements and we show three typical patterns below.

$$\begin{array}{ccc}
 \overline{\overline{1 \ 2 \ 3}} & \rightarrow & \overline{\overline{1 \ 2 \ 3}} \\
 \overline{\overline{1 \ 2 \ 3 \ 4}} & \rightarrow & \overline{\overline{1 \ 2 \ 3 \ 4}} \\
 \overline{\overline{1 \ 2 \ 3 \ 4}} & \rightarrow & \overline{\overline{1 \ 2 \ 3 \ 4}}
 \end{array}$$

Longer intervals have to take into account the tree structure and the effects can be worked out by the reader.

What is discovered by checking the four possible rotations of each of the three trees in the left column of (11) is that a rotation either produces another tree in the left column or a tree in a face specified by (10). Similarly a rotation of a tree in the right column of (11) either produces another tree in the right column or a tree in a face specified by (10). For example in any of the trees in the left column of (11), a rotation using top vertex results in the shadow interval $[1, 5]$ which is in (10). Similarly in the right column one rotation using the top vertex in each tree produces $[1, 5]$ while the other rotation using the top vertex produces $[3, 6]$ for the first tree and $[4, 6]$ for the second and third trees. Other rotations are left to the reader.

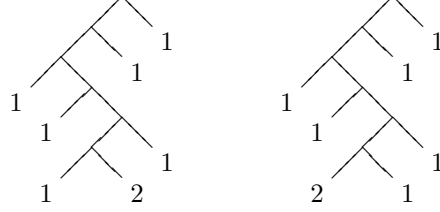
It follows that the trees in the left column of (11) are in a different component of complement of the union of the faces given by (10) from the trees in the right column of (11), and thus the union of faces specified by (10) separates the 1-skeleton of A_4 .

10.3. Long paths. Consider the color vector (written without commas) $\mathbf{c} = 1^m 21^n$ with m and n at least zero. This vector colors vertices of A_d with $d = m + n - 1$. We know that the diameter of the 1-skeleton of A_d is no more than $2d - 4$ or $2m + 2n - 5$. Thus there is a path (that ignores signs) in the 1-skeleton of length no more than $2m + 2n - 5$ between any two vertices of A_d . However, we will show that there are vertices for which \mathbf{c} is valid for which the shortest path between them that is sign consistent (lies in the color graph of \mathbf{c}) has length mn .

We first note that any tree T for which \mathbf{c} is valid must be a vine. Any exposed caret in T must use the unique 2 in \mathbf{c} as a color of a leaf edge, and this leaf edge can only be in one caret.

All carets in T other than the exposed caret will have one descending edge as an internal edge and the other descending edge will be a leaf edge and under \mathbf{c} , this leaf edge will be colored 1. Thus the other (internal) descending edge of the caret will be colored 2 or 3. Note that even the exposed caret of T has exactly one descending edge colored 1. If a caret in T has its left descending edge colored 1, we will label that caret l and if its right descending edge is colored 1, we will label that caret r . Since T is a vine, there is a well defined top-to-bottom ordering of the carets, and we can read the labels r or l in order from top to bottom. This gives is

a finite word in the alphabet $\{l, r\}$. For example, in the trees below,



the left tree gives the word $rrllrl$ and the right tree gives the word $rrllrr$. Note that the two trees are identical, but the words and the color vectors are different. The labeling convention we use disagrees with that in Section 3 of [5].

The following can easily be verified by the reader.

- (1) The colors of the internal edges alternate between 3 and 2 starting with 3 at the lowest internal edge.
- (2) Two adjacent carets have the same sign if and only if they have opposite labels.
- (3) Any sign consistent rotation changes a single appearance of rl to lr in the word for the tree (or the reverse) and leaves the rest of the labels the same.
- (4) If a word w in $\{l, r\}$ is given by a vine T colored by $\mathbf{c} = 1^m 21^n$, then the number of appearances of l in w is m and the number of appearances of r in w is n .
- (5) The vines that can be colored by $\mathbf{c} = 1^m 21^n$ are in one-to-one correspondence with the words in $\{l, r\}$ that use m copies of l and n copies of r .
- (6) If w_1 and w_2 are two words in $\{l, r\}$ given by vines V_1 and V_2 , respectively, colored by $\mathbf{c} = 1^n 21^n$, then the shortest sign consistent path (path in the color graph of \mathbf{c}) from V_1 to V_2 is the number of lr to rl or reverse moves needed to take w_1 to w_2 .
- (7) If V_1 gives the word $w_1 = l^m r^n$ and V_2 gives the word $w_2 = r^n l^m$ when colored by $\mathbf{c} = 1^m 21^n$, then the shortest sign consistent path (path in the color graph of \mathbf{c}) from V_1 to V_2 as colored by \mathbf{c} is of length mn . Further there is such a path of that length. (The vines V_1 and V_2 in are among those covered by Proposition 9 of [5].)

These combine to give the following.

Proposition 10.2. *The diameter of the color graph of $\mathbf{c} = 1^m 21^n$ is mn .*

Note that for $m = n$, we get a coloring of A_{2n} and a color graph of diameter n^2 while the (uncolored) diameter of the 1-skeleton of A_{2n} is no more than $4n - 4$.

11. SIGN STRUCTURES

In this section we build a signed graph $\Sigma(w)$ for each edge path w in an associahedron. We will show that the path w is “valid” in that there is a finite, binary tree T and a sign assignment for T so that w is valid for T in the sense of Proposition 6.2 if and only if $\Sigma(w)$ is balanced. This will verify the claim made after Proposition 6.2.

The vertex set of our graph $\Sigma(w)$ will be the internal vertices of the tree T above. In [4], Carpentier develops a similar but not identical criterion for the validity of a path. In particular, the vertex set of the two structures is different with the vertices used in [4] being the elements of the path w and not the vertices of the tree T .

11.1. The signed graph. Let w be an edge path in an associahedron that starts at a vertex D and ends at a vertex R . We can think of w as a word in the symmetric generators. Let this word be $w = [v_1][v_2] \cdots [v_k]$. Thinking of each $[v_i]$ and (D, R) as elements of F , let us write

$$w = [v_1][v_2] \cdots [v_k] = (D, R).$$

However, there are other paths that represent the same element of F . These paths either have the same endpoints but get from one to the other by a different route, or the paths read as the same string of symmetric generators but connect different pairs of vertices, or both. For example, in the drawing (6) of A_3 there are four different paths that read as $[u][u1]$. Our definition of $\Sigma(w)$ will depend on the word in the symmetric generators and not on the particular start and end vertices.

The graph $\Sigma(w)$ will be an undirected, signed graph with one edge for each edge in the path w . The graph $\Sigma(w)$ may have parallel edges but will have no loops. The vertex set will start out as the set of vertices in \mathcal{T} so we will simply use \mathcal{T} to denote the vertex set. This is an infinite set and is more than needed. After proving a key lemma, we will cut down the set of vertices.

We define $\Sigma(w)$ inductively and if $w = [v_1][v_2] \cdots [v_k]$, then we will need $p_i = [v_1][v_2] \cdots [v_i]$ where $1 \leq i \leq k$ and we will need p_0 to be the identity in F . We start with $\Sigma_0 = \Sigma(p_0)$ as the graph on \mathcal{T} with no edges.

If $\Sigma_{i-1} = \Sigma(p_{i-1})$ is defined, then we form Σ_i by adding an edge e_i to Σ_{i-1} . We need to pick the endpoints of e_i and its sign. Let x and y be the pivot vertices of $[v_i]$. The endpoints of e_i will be $s = (x)p_{i-1}^{-1}$ and $t = (y)p_{i-1}^{-1}$. Let d_s and d_t be the degrees of s and t , respectively, in Σ_{i-1} . Then e_i will be given a positive sign if and only if $d_s + d_t$ is even. The graph $\Sigma(w)$ that we seek is Σ_k . Note that the starting vertex of w is not relevant to the definition.

Rather than explain this definition, we will state and prove the following. A signed graph is *balanced* if every closed walk traverses an even number of negative edges.

Theorem 11.1. *Let w be an edge path in an associahedron that starts at a vertex D . Then*

- (1) *all of the endpoints of edges in $\Sigma(w)$ are internal vertices of D , and*
- (2) *there is a sign assignment for D for which w is valid if and only if $\Sigma(w)$ is balanced.*

Proof. If the p_i are as defined above, then Dp_i is the i -th vertex visited by w with D as the 0-th. Lemma 7.1 makes p_i a bijection from the internal vertices of D to the internal vertices of Dp_i . Item (1) follows immediately.

If there is a sign assignment for D for which w is valid, then to show that $\Sigma(w)$ is balanced it is sufficient to show that each edge in $\Sigma(w)$ is positive if and only if it connects two vertices in D with the same sign.

The edge e_i in the construction above connects x and y where $x p_{i-1}$ and $y p_{i-1}$ are the pivot vertices of $[v_i]$. But the sign of $x p_{i-1}$ in the sign assignment of Dp_{i-1} is the sign of x in D as modified by $p_{i-1} = [v_1][v_2] \cdots [v_{i-1}]$. The modification consists one negation for each $x p_j$ with $1 \leq j < i$ for which $x p_j$ is a pivot vertex of $[v_{j+1}]$. But this is exactly the degree of the vertex x in Σ_{i-1} . Similarly the sign of $y p_{i-1}$ in Dp_{i-1} is the sign of y negated a number of times which is the degree of the vertex y in Σ_{i-1} . Thus the sign of e_i is positive if and only if the signs $x p_{i-1}$ and $y p_{i-1}$ in Dp_{i-1} both agree or both disagree, respectively, with the signs of x

and y in D . The validity of w says that the signs of $x p_{i-1}$ and $y p_{i-1}$ must be equal. Thus the sign of e_i is positive if and only if the signs of x and y in D are equal. This proves the desired condition needed to show that $\Sigma(w)$ is balanced.

If it is known that $\Sigma(w)$ is balanced, then there is a sign assignment of the internal vertices of D with the property that each edge in $\Sigma(w)$ is positive if and only if it connects two vertices in D with the same sign. Considerations almost identical to the previous paragraph show that w is valid for this sign assignment on D . \square

The details in the above proof give us the following. If $\Sigma(w)$ is balanced (continuing the notation of this section), then a sign assignment of the internal vertices of D is *compatible* with $\Sigma(w)$ if an edge in $\Sigma(w)$ is positive if and only if it connects edges with the same sign. We say that a coloring of the pair (D, Dw) is *compatible* with $\Sigma(w)$ if the associated sign assignment on the internal vertices of D is compatible with $\Sigma(w)$.

Theorem 11.2. *Let w be an edge path in an associahedron that starts at a vertex D . Then the following hold.*

- (1) *The path w is valid with a sign assignment for D if and only if the sign assignment is compatible with $\Sigma(w)$.*
- (2) *The number of sign assignments for which w is valid is 2^p where p is the number of components of $\Sigma(w)$ regarded as a graph on the internal vertices of D .*
- (3) *The number of valid colorings of the pair (D, Dw) that are compatible with $\Sigma(w)$ modulo permutations of the colors is 2^{p-1} with p as in (2).*

Proof. We have (1) from the details of the proof above, (2) is immediate, and (3) from the fact that there is one representative in each permutation class that has root color 1 and the sign of the child of the root of D positive. \square

Note that a single vertex can be a component of $\Sigma(w)$.

For an edge path w in an associahedron, we call $\Sigma(w)$ the *sign structure* of w .

We say that an edge path w in an associahedron is *sign consistent* if $\Sigma(w)$ is balanced.

We have one observation that can be made immediately about sign consistent paths.

Lemma 11.3. *A subpath of a sign consistent path is sign consistent.*

Proof. If a sign consistent path $w = pvs$ starts at tree D with a given sign assignment σ , then the subpath v starts at $(D^\sigma)p$ and must be consistent with the sign assignment there. \square

11.2. The second signed path conjecture. We can apply Theorem 11.1 to give a statement equivalent to Conjecture 6.4.

Conjecture 11.4. *For every pair of finite, binary trees (D, R) with the same number of leaves, there is a sign consistent path from D to R .*

11.3. The vertex set of a sign structure. Giving the set of vertices of \mathcal{T} to $\Sigma(w)$ is clearly excessive, and also unrevealing. Item (1) of Theorem 11.1 hints that if w is a path starting at a tree D , then the internal vertices of D might make a good vertex set for $\Sigma(w)$. However, we have pointed out that different paths can read

as the same string of rotation symbols, and the edges of $\Sigma(w)$ really only depend on the string of symbols. Another obvious choice would be the set of endpoints of the edges of $\Sigma(w)$. For reasons that we hope will be made clear, we will adopt a compromise.

Let w be a path in some A_d , and let S be the set of endpoints of $\Sigma(w)$. There is a smallest tree T containing all of S among the internal vertices of T . We let the internal vertices of T be the set of vertices of $\Sigma(w)$.

The reason for our choice of vertex set is the next result.

Theorem 11.5. *Let w be an edge path in an associahedron, and let T be the tree whose internal vertices are the vertices of $\Sigma(w)$. If the pair (T, Tw) is prime, then $\Sigma(w)$ is connected.*

Proof. The proof will be inductive, and we will have to prove more than the statement of the theorem.

Let w have length n , and let p_i be the prefix of length i of w , $0 \leq i < n$. Obviously, p_0 is the empty word. Recall that we can think of w and all the p_i as functions, where the special case of $i = 0$ has p_0 as the identity.

We need a preliminary discussion before giving the statements that we will prove. If S is a tree and S' and S'' are subtrees of a tree S having disjoint non-empty sets of internal vertices and there is an edge from an internal vertex of S' to an internal vertex of S'' , then either a leaf of S' is an internal vertex of S'' or the reverse. The cases are mutually exclusive. In both cases, we say that S' and S'' are adjacent. In the case that a leaf of S' is an internal vertex of S'' , we say that S' is above S'' . Otherwise we say that S'' is above S' .

Consider the following statements where $1 \leq i \leq n$.

α_i : If C is a component of $\Sigma(p_i)$ and $V(C)$ is its vertex set, then for each j with $0 \leq j \leq i$, there is a subtree $T_{C(j)}$ of Tw_j whose internal vertices are exactly the vertices $(V(C))p_j$.

β_i : Let C and D be components of $\Sigma(p_i)$. If $T_{C(0)}$ and $T_{D(0)}$ are adjacent with $T_{C(0)}$ above $T_{D(0)}$, then for each j with $0 \leq j \leq i$, the subtrees $T_{C(j)}$ and $T_{D(j)}$ are adjacent with $T_{C(j)}$ above $T_{D(j)}$. If $T_{C(0)}$ and $T_{D(0)}$ are not adjacent, then for each j with $0 \leq j \leq i$, the subtrees $T_{C(j)}$ and $T_{D(j)}$ are not adjacent.

We have that α_1 is trivial, and we have that β_1 follows from the nature of a single rotation.

We assume α_k and β_k hold for some $k < n$. Let $[u] = w_{k+1}$. There are two cases to consider.

Case I: Both pivot vertices of $[u]$ are internal vertices of a single $T_{C(k)}$. In this case, the edge added to $\Sigma(p_k)$ to create $\Sigma(p_{k+1})$ has both of endpoints in the vertices of a single component C of $\Sigma(p_k)$. Thus the components of $\Sigma(p_k)$ and $\Sigma(p_{k+1})$ have identical vertex sets and α_{k+1} holds for $1 \leq j \leq k$. To get α_{k+1} for $j = k + 1$, we note that the vertex set $(V(C))p_{k+1} = ((V(C))p_k)[u]$ is the image under $[u]$ of the set of internal vertices of the subtree $T_{C(k)}$. From the illustration in (4), we see that this forms the set of internal vertices of a subtree of Tw_k . All other subtrees corresponding to components of the sign structure are carried to Tw_{k+1} isomorphically. This proves α_{k+1} .

The statement β_{k+1} clearly holds for $1 \leq j \leq k$. The β family of statements says that the relation “adjacent to” and the relation “over” among the relevant subtrees are preserved by the prefixes of w . The illustration in (4) also shows that these relations are preserved by a single rotation. Thus β_{k+1} also holds for $j = k + 1$.

Case II: The pivot vertices of $[u]$ are internal vertices of two different $T_{C(k)}$ and $T_{D(k)}$. Since the pivot vertices are endpoints of an edge, we have that $T_{C(k)}$ and $T_{D(k)}$ are adjacent, and we can assume that $T_{C(k)}$ is over $T_{D(k)}$. Since β_k says that the relation “adjacent to” and the relation “over” are preserved by the p_j with $1 \leq j \leq k$, we know that T_C and T_D are adjacent with T_C over T_D . Similar statements apply to the $T_{C(j)}$ and $T_{D(j)}$ for $1 \leq j < k$, while for $j = k$ it is our hypothesis. It is now seen that for $0 \leq j \leq k$, the union of $T_{C(j)}$ with $T_{D(j)}$ is a subtree $T_{A(j)}$ of Tp_j whose set of internal vertices is the vertex set of a component of $(\Sigma(p_{k+1}))p_j$. Thus if we replace $T_{C(0)}$ and $T_{D(0)}$ by $T_{A(0)}$ in T and keep all the other subtrees the same, then we now have the desired one-to-one correspondence between subtree and component of $\Sigma(p_{k+1})$. This is carried in the correct way to each Tp_j by prefixes p_j for $1 \leq j \leq k$, and the argument that this works for $j = k + 1$ follows because we are now in the situation of Case I and the argument for $j = k + 1$ for both α and β applies here.

We now assume that $\Sigma(w) = \Sigma(p_n)$ is not connected. Let C be a component of $\Sigma(w)$ so that T_C is minimal with respect to the “over” relation. This will make every leaf of T_C a leaf of T . Thus every leaf of $T_{C(n)}$ is a leaf of Tw . Since rotations, and thus chains of rotations, preserve prefix order and thus the left-right order of the leaves, the leaves of the subtrees T_C and $T_{C(n)}$ define exactly the same intervals in the leaf numberings of T and Tw . Since $\Sigma(w)$ is not connected, this interval is not the entire interval of leaf numbers. Thus (T, Tw) is not prime. \square

The following corollary combines primality and consistency with Theorem 11.5.

Corollary 11.5.1. *If (T, Tw) is a prime pair with w a consistent edge path starting at T , then there is a unique normal coloring for T compatible with $\Sigma(w)$ with the sign of \emptyset positive.*

We next enter a somewhat speculative discussion on when two paths might give the same coloring.

11.4. Relations among the paths. This section contains a set of related observations about the sign structures. We give them to explain the wording of some conjectures that we raise about colorings that arise from sign structure considerations.

11.4.1. Moving paths across faces. If w and w' are two edge paths in some A_d with the same endpoints, then as words in the symmetric generators, they represent the same element of F . It follows from the presentation discussed in Section 7.6 that as words, we can alter w using a sequence of “square” or “pentagonal” relations. We will call these alterations “square” or “pentagonal” moves. These moves, together with cancellation or introduction of consecutive inverse pairs, can be applied to w so that the result equals w' . From the nature of the square and pentagonal moves, it is clear that these moves can be realized as moves across 2-dimensional faces in A_d . We look at the two kinds of alterations.

11.4.2. Moving paths across pentagons. Let D and R be two non-adjacent vertices in A_2 . See (5). There are two simple paths between them—one of length three and one of length two. The path of length three is always inconsistent and the path of length two is always consistent. This is easy enough to check by hand. For example, $[u0][u][u1]$ is inconsistent, but $[u0][u]$ is consistent. Because of the

action of the dihedral group of order ten on A_2 , it is only necessary to check one of each length. Thus a pentagonal move is capable of changing a path from consistent to inconsistent or vice versa.

11.4.3. *Adding or removing canceling pairs.* It is also clear that a consistent path can be “ruined” by inserting consecutive canceling pairs. The consistent path $[u0][u]$ represents the same element of F as the inconsistent path $[u0][u][u1][\overline{u1}]$. We will comment again on the problems introduced by increasing the length of a path.

11.4.4. *Moving paths across squares.* There are two cases to consider.

In the simpler case, a two-edge subpath along part of a square is replaced by the other two-edge subpath along the other part of the square. If the various square relations are investigated, it is found that the first two-edge subpath introduces two edges in Σ of the path that have disjoint endpoints and that the other two-edge subpath introduces the same two edges in Σ in the other order. This is in spite of the fact that the four edges around the square correspond to three different symmetric generators. When two edges with disjoint end points are introduced in succession into Σ , reversing the order does not change any of the sign assignments in Σ . Thus a two-edge to two-edge move across a square preserves sign consistency.

In the more complicated case, a three-edge subpath is replaced by a one-edge subpath. The three edge path introduces three edges into Σ in which two are parallel and one has endpoints disjoint from the parallel edges. The two parallel edges acquire the same signs as each other. The shorter path leaves out the two parallel edges. Removing a pair of parallel edges does not alter the parities of any vertex. Thus it is seen that a three-edge to one-edge move across a square does not destroy sign consistency, but it might convert a sign inconsistent path into a sign consistent path by removing a damaging term. For example $w_1 = [u0][u][u1][u111][\overline{u1}]$ is sign inconsistent. But this represents the same element of F and of E as $w_2 = [u0][u][u11]$ which is sign consistent.

Since the passage from w_2 to w_1 takes a shorter consistent path to a longer inconsistent path, we see again that increasing the length of a path can cause problems with consistency.

11.4.5. *Shortest paths are not always the best.* We give an example here to show that making paths shorter is not always the best. The path

$$[u][u1]^3[u]^{-1}$$

is easily shown to be sign inconsistent. However this represents the same element of F as

$$([u0]^{-1}[u])^3$$

which is longer and sign consistent. Computer search verifies that no word with less than six symbols is sign consistent and equivalent to the words above. For the interest of the reader we mention that there are 5 different paths of length 6 equivalent to the words above with 5 different sign structures among them.

11.4.6. *Parallel edges in the sign structure.* The discussion in Section 11.4.4 shows that parallel edges can come from canceling pairs and conjugations. However, they do not have to arise that way.

The word $[u][u][\overline{u1}]$ has a sign structure with two parallel edges that have opposite signs and is thus clearly not sign consistent. It is also one of the forbidden three-edge paths in A_2 . See (5).

The word $[u][u][u1][\overline{u11}]$, has parallel edges with the same sign and is sign consistent. It cannot be made shorter. With $u = \emptyset$, the reduced tree pair for this element of F is

$$(12) \quad \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} , \quad \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \right).$$

These can be located in the drawing (6) of A_3 where it is seen that the shortest edge path between them is of length 4. It is an exercise (which we leave to the reader) to show that if two vertices lie in a face of an associahedron, then the shortest edge path between them lies in that face. The exercise consists of looking at the set of subtrees that define that face and noting that all rotations outside that set of subtrees will add excessively to the length of the path. It follows, that no shorter word in the symmetric generators represents the element of F pictured in (12).

It should be noted that the element in (12) is one of the smallest elements that has a rigid coloring. It is easy to check that positive normal coloring consistent with the word $[u][u][u1][\overline{u11}]$ comes from the color vector $(2, 2, 3, 1, 3)$. It is just as easy to check that the positive rigid coloring for the pair comes from the color vector $(2, 2, 1, 3, 3)$.

The observations that we have made lead to the following.

11.5. Two conjectures. From the remarks in Section 11.4, moves across single pentagons have drastic consequences. However, it is possible to find moves across pairs of pentagons that take sign consistent paths to sign consistent paths. Experiment hints that the resulting pairs of paths have drastically different sign structures, leading to different colorings of the two trees at the endpoints.

Moves across squares tend to be less drastic, although careless applications can introduce inconsistencies in the paths. If we restrict moves to square moves, then we create *square path classes*, by declaring that two paths are *square equivalent* if one can be carried to the other (without changing the endpoints) by only making square moves. The square path classes all lie within a single element of E .

We now give two conjectures.

Conjecture 11.6 (Well definedness). *Let (T, S) be prime and let w and w' be sign consistent paths from T to S that are square equivalent with no shorter path in the square equivalence class that is also sign consistent. Then $\Sigma(w) = \Sigma(w')$.*

Conjecture 11.7 (Injectivity). *Let (T, S) be prime and let w and w' be sign consistent paths from T to S with no shorter path in their respective square equivalence class that is also sign consistent. If $\Sigma(w) = \Sigma(w')$, then w and w' represent the same element of E .*

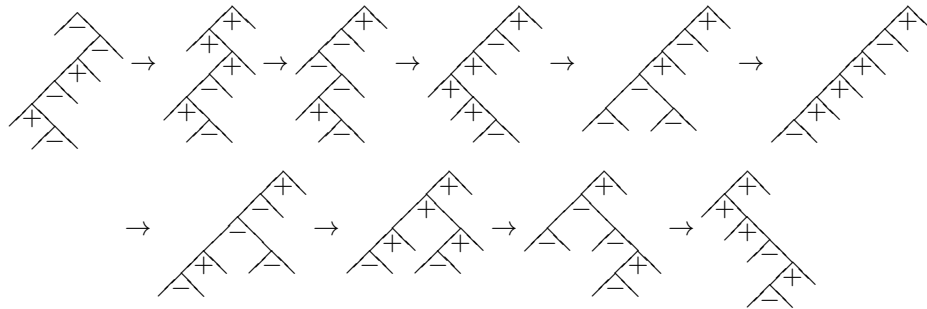
If the first conjecture above holds, then each element of E with a sign consistent representative corresponds to a unique sign structure. If the image of E in F is prime, then this gives a single well defined coloring of the reduced pair representing the element of F . If the second conjecture above holds, then no two elements of E that give colorings (have sign consistent representatives) can give the same coloring. It follows from Theorem 6.3 that every flexible coloring comes from an element of E .

11.6. On the injectivity conjecture. What would be desirable now would be for elements of E expressed as words in the symmetric generators to have a nice normal form. The elements of F expressed as words in the generating set $\{1^n \mid n \geq 0\}$ have a nice normal form. See [3]. Were elements of E to have a nice normal form, then two elements of E reduced to a normal form and still different might be in good enough form to be able to show that where they differ forces a difference in their respective sign structures. Alas, the relations in E among the symmetric generators allow for a lot of moving around and a nice normal form seems elusive. This approach to the injectivity conjectures still seems to have promise.

11.6.1. Comment. It is possible to show that if $\Sigma(w)$ is not connected, then w is square equivalent to a path that breaks into a finite number of subpaths in one-to-one correspondence with the components of $\Sigma(w)$ where each subpath generates its corresponding component.

11.7. On the converse to Theorem 11.5. We use the notation of Section 10.3. The color vector 1332111 is valid for the pair of vines described by $lrrrl$ and $rlllr$. The reader will note that this is not prime and that the “prime factors” are a pair of trees with four carets each together with the pair of 2-caret trees used in $[\emptyset]$. If the colors are applied and the signs of the internal vertices are computed, then it will be noted that the four caret pair is rigidly colored (it is among the smallest non-identity examples and one of the standard generators of F_4) and that the two caret pair is flexibly colored.

The shortest path (computer search and easily verified by an exhaustive search by hand) connecting the two colored trees has nine steps. Its sign structure is connected. Among the features of the sign structure are two pairs of parallel edges, two independent cycles (treating the parallel edges as one) and only one vertex not part of a cycle. The example is a good illustration of the “luck” demanded by the Signed Path Conjectures. The path is illustrated below. The reader can verify that the color vector 1332111 produces the signs shown.



The flexible coloring of the top two carets seems to “corrupt” the rigidity of the bottom four. The process that does this is rather complicated. To see the extent of the ability of a small amount of flexibility to corrupt rigidity, the reader can supply the pleasant inductive step for the following. The induction is on the number of carets. A 4-tree is a finite, binary tree in which every leaf is of even level. The reason for the terminology is obvious once the reader tries to draw one. It is an exercise that every element of F_4 can be represented by a pair of 4-trees. Another exercise is that a positive, rigid color vector for a 4-tree always has the form $(132)^j 1$.

Lemma 11.8. *Let T be a 4-tree and let A be the trivial tree. In $T \hat{\wedge} A$, give \emptyset the positive sign and let the signs of the rest of the vertices agree with the positive, rigid coloring on T (so that the vertex 0 has positive sign in $T \hat{\wedge} A$). Then there is a sign consistent path from $T \hat{\wedge} A$ to the right vine in which all internal vertices are signed positively.*

11.8. Non-prime maps. The relation between the prime factors of a map and the components of a sign structure are not clear. From the above example, it is seen that there is no one-to-one correspondence. From the discussion leading to (1) in Section 1.1.1, we can derive colorings of a non-prime map from the colorings of the prime factors and compute the number of colorings of the non-prime map from the number of colorings of the prime factors. However, we cannot always compute the components of a sign structure by simple knowledge of the prime factors of a tree pair.

Question 11.9. *What is the relationship between the prime factors of a map and the components of the sign structure?*

12. ACCEPTABLE COLOR VECTORS

This section and the sections that follow gather observations that either fill earlier promises, or give extra facts, some of which are related to each other, and that are somewhat off the main narrative. Section 15 counts many of the objects that have been encountered.

After Lemma 5.3, we announced that a color vector is acceptable (valid for some tree) if and only if the vector is not a constant and does not sum to zero. Here we prove that claim.

A color vector that is constant produces a zero wherever an exposed caret exists, and a vector that sums to zero produces a zero at the root edge. Thus the conditions are necessary. We give the result after a sequence of lemmas.

We will use \mathbf{v} to denote a color vector that is not a constant and does not sum to zero. We will represent the contents of a vector as strings and combine vectors by concatenation. Letters x , y and z will represent unspecified colors that are assumed to be different if the letters are different.

Lemma 12.1. *A constant vector x^i sums to zero if i is even and x if i is odd.*

Lemma 12.2. *The vectors $x^n y$ and xy^n are always acceptable.*

Proof. The first is valid for a right vine (see Section 8.4), and the second is valid for a left vine. \square

Lemma 12.3. *If $\mathbf{v} = \mathbf{ps}$ and both \mathbf{p} and \mathbf{s} are acceptable, then \mathbf{v} is acceptable.*

Proof. Since the sum of \mathbf{v} is not zero, the sums of \mathbf{p} and \mathbf{s} are not equal. If \mathbf{p} is valid for A and \mathbf{s} is valid for B , then \mathbf{ps} is valid for $A \hat{\wedge} B$ (see Section 3.6). \square

Lemma 12.4. *If $\mathbf{v} = x^i \mathbf{s}$ or $\mathbf{v} = \mathbf{s} x^i$, $i \geq 1$ and \mathbf{s} is acceptable with sum different from x , then \mathbf{v} is acceptable.*

Proof. If \mathbf{s} is valid for A , then $x^i \mathbf{s}$ is valid for the tree formed by identifying the root edge of A with the rightmost leaf edge of the right vine V_i (right vine with i internal vertices). A similar construction with a left vine handles $\mathbf{s} x^i$. \square

The next lemma is not used, but is too cute to omit.

Lemma 12.5. *If $\mathbf{v} = x^i y^j$, $i \geq 1$, $j \geq 1$, then \mathbf{v} is acceptable.*

Proof. The sum of \mathbf{v} is not zero, so at least one of i or j is odd. Assume i is odd. From Lemma 12.2, we can assume $j > 1$. The sum of $x^i y$ is z , and by Lemma 12.4, $\mathbf{v} = (x^i y) y^{j-1}$ is acceptable. If j is odd, we assume $i > 1$ and a similar proof works. \square

Recall that $[xyz]$ denotes an unspecified choice of one of x , y or z . The expression $[xyz]^n$ denotes a string of n such choices rather than one choice repeated n times.

Proposition 12.6. *A color vector of length at least 2 is acceptable if and only if it is non-constant and has a non-zero sum.*

Proof. We only need to argue one direction and we assume \mathbf{v} is not constant and does not sum to zero. The claim is true if \mathbf{v} has length 2.

We know $\mathbf{v} = x^i y [xyz]^{n-i}$ with $i \geq 1$. If the sum of \mathbf{v} is y or z , then $\mathbf{v} = x\mathbf{s}$ where the sum of \mathbf{s} cannot be zero and cannot be x . Now \mathbf{v} is acceptable by Lemma 12.4.

If the sum of \mathbf{v} is x and the last color in \mathbf{v} is not x , then $\mathbf{v} = \mathbf{p}[yz]$ and where the sum of \mathbf{p} is x minus the last color in \mathbf{v} . Again \mathbf{v} is acceptable by Lemma 12.4.

If the sum of \mathbf{v} is x and the last color in \mathbf{v} is x , then $\mathbf{v} = x(x^{i-1} y [xyz]^{n-i-1})x$, and the sum of $\mathbf{m} = (x^{i-1} y [xyz]^{n-i-1})$ must be x . Now $\mathbf{m} = \mathbf{p}\mathbf{s}$ where $\mathbf{p} = x^{i-1} y$ and \mathbf{s} is the rest of \mathbf{m} . The sum of \mathbf{p} is y or z , so the sum of \mathbf{s} is z or y . Now $\mathbf{v} = (x\mathbf{p})(\mathbf{s}x)$ has been decomposed into two pieces with lengths at least two whose sums are each different from x . Since each includes x , neither is constant. By induction on length, each of $x\mathbf{p}$ and $\mathbf{s}x$ are acceptable and \mathbf{v} is acceptable by Lemma 12.3. \square

Proposition 12.6 supplies a converse to Proposition 2 of [5]. To state the converse, we write that if w is a word in the non-identity elements of $\mathbf{Z}_2 \times \mathbf{Z}_2$ and (x, y, z) represent these three elements in some order, then $|w|_x$ is the number of occurrences of x in w , and so forth.

Lemma 12.7. *If w is a non-constant finite word in (x, y, z) and*

$$|w|_x \equiv |w|_y \not\equiv |w|_z \equiv |w| \pmod{2},$$

then w is valid for some tree T .

Proof. All that is needed is that the sum not be zero. The brief proof that the sum is z is given in [5] from its point of view. From our view the argument is as follows. If $|w|_x$ and $|w|_y$ are both even, then the letters x and y contribute zero to the sum and $|w|_z$ being odd makes the letter z contribute z . If $|w|_x$ and $|w|_y$ are both odd, then letters x and y contribute $x + y = z$ and the letter z contributes zero. \square

13. PATTERNS

13.1. Patterns and multiplication. This section generalizes some of the observations in the proof of Proposition 9.2 which found a group arising from a certain sign assignment on \mathcal{T} . However, we have not explored the generalization that we are about to present and it is not clear that it opens up much in the way of examples. There is one example that is rather trivial to establish and that will be given later in this section. Even though it is trivial to establish, it is not trivial in structure. However, the structure has not yet been explored.

A *pattern* P is a sign assignment on \mathcal{T} . Any finite, binary tree inherits a sign assignment from P , and the convention that the root edge be colored 1 gives each finite binary tree an edge coloring derived from P . If the vector of the colors of the leaf edges of two trees turn out to be identical under the coloring derived from P , then the common color vector is valid for the pair. In such cases, we will say that the pair is P -compatible.

An example of a pattern is the positive rigid pattern that we denote P^r . It was defined as $P^r(v) = (-1)^{|v|}$. The negative rigid pattern $-P^r$ was also defined. Proposition 9.2 shows that the P^r -compatible pairs form a subgroup of F . We discuss conditions under which an arbitrary pattern P leads to a subgroup.

To help with the discussion, we look at subpatterns. If P is a pattern and u is a vertex in \mathcal{T} , then P_u will be defined by $P_u(v) = P(uv)$. If we recall that \mathcal{T}_u is defined by $\mathcal{T}_u = \{uv \mid v \in \mathcal{T}\}$, then we see that P_u is the composition of the obvious isomorphism $v \mapsto uv$ from \mathcal{T} to \mathcal{T}_u with the restriction of P to \mathcal{T}_u .

If we now have a pattern P and a pair of P -compatible pairs (A, B) and (C, D) , then $(A, B)(C, D)$ has a coloring since for trivial reasons the coloring of (A, B) and (C, D) induce the same coloring on $B \cap C$ and the Compatibility Lemma (Lemma 8.3) applies. However, the coloring on the product might not be derived from P .

We say that the pair of pairs $((A, B), (C, D))$ is P -compatible if both of the following hold.

- (1) For every leaf v of $B \cap C$ that is a leaf of C and not a leaf of B and leaf w of D that occupies the same position in the left-right order of the leaves of D as v does in the leaves of C , then $P_v = P_w$.
- (2) For every leaf v of $B \cap C$ that is a leaf of B and not a leaf of C and leaf w of A that occupies the same position in the left-right order of the leaves of A as v does in the leaves of B , then $P_v = P_w$.

Our overuse of the term P -compatible leads to following which has to be read carefully.

Lemma 13.1. *If P is a pattern and $((A, B), (C, D))$ is a P -compatible pair of P -compatible pairs of binary trees (sic), then the product $(A, B)(C, D)$ is P -compatible.*

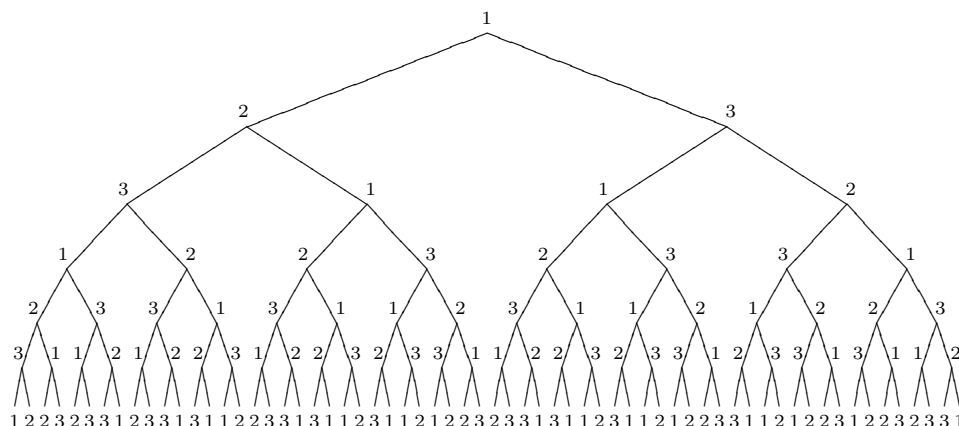
Proof. The proof is identical to the proof of Proposition 9.2. □

13.2. The positive pattern and group. We apply Lemma 13.1 to a trivial setting. We define the pattern P^+ by having $P^+(v)$ positive for all v . For every vertex u , we have $P_u^+ = P^+$.

Lemma 13.2. *The set of P^+ -compatible pairs of trees forms a group under multiplication.*

Proof. There is no way for the hypotheses of Lemma 13.1 to fail. □

Here is what the all positive pattern P^+ looks like to level 6.



Let us use F^+ to denote the subgroup of F consisting of the P^+ -compatible pairs. While it is clear that the identity element is in F^+ , it takes work to find non-trivial elements. Below is what seems to be the simplest (measured by the size of the trees involved).

$$\left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \quad \diagdown \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \end{array} \right), \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \quad \diagdown \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \end{array} \right)$$

It has a nice symmetry that other examples do not. Powers of this element follow a nice pattern. A less symmetric example is shown below.

$$\left(\begin{array}{c} \diagup \\ \diagdown \end{array} , \begin{array}{c} \diagup \\ \diagdown \end{array} \right)$$

The group F^+ has a property shared by F . If T is a finite binary tree, if (A, B) is a pair in F^+ , and v is a leaf of T , then we can form a new element of F^+ . We do this by attaching A to T at v to form A' and B to T at v to form B' . This means that A' is obtained by identifying the root edge of A with the leaf edge of T that impinges on v . A more technical definition is $A' = T \cup vA$ where the vertices of vA are those in $\{vu \mid u \in A\}$. The new element of F^+ is (A', B') .

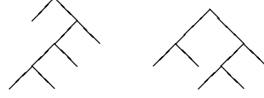
Note that the element of F^+ that results depends only on (A, B) and v and not on the particular T that has v as a leaf. Following [16], we refer to (A', B') as the *deferment* of (A, B) to v . We thus have that, like F , the group F^+ is closed under deferment.

13.3. Neighborhoods of positive colorings. Each vertex in A_d has d edges impinging on it. For a given color vector \mathbf{c} , there might be fewer than d valid edges impinging on a given vertex. If the sign assignment on a given vertex is all positive (or all negative), then all edges leaving from that vertex are valid. It might be guessed that starting with a coloring of a vertex that gives an all positive sign assignment would give a large color graph. To give words to the concept, let us use the word *positive neighborhood* of a vertex T of A_d to refer to the color graph of that vector \mathbf{c} (normalized to give 1 as the root edge color) that gives the all positive sign assignment on T .

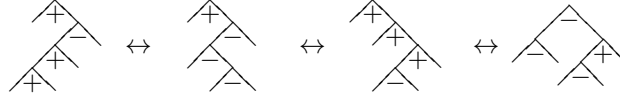
In A_2 , the positive neighborhoods are all the paths of length 2. Since every pair of vertices in A_2 is in a path of length 2, we have that there is a coloring for every pair of vertices in A_2 .

If for every d , it is true that each pair of vertices in A_d lies in the positive neighborhood of some third vertex, then the 4CT would follow. In fact this fails for $d = 3$.

The two trees shown below



have only one normalized coloring and it is given by the color vector 21211. With the top vertex of the left tree assigned $+$, there are only 8 possible sign assignments to check. These produce 8 color vectors for the left tree, of which the only one valid for the right tree is 21211. The full color graph for the vector 21211 is shown below where the signs of the vertices are shown rather than the colors.



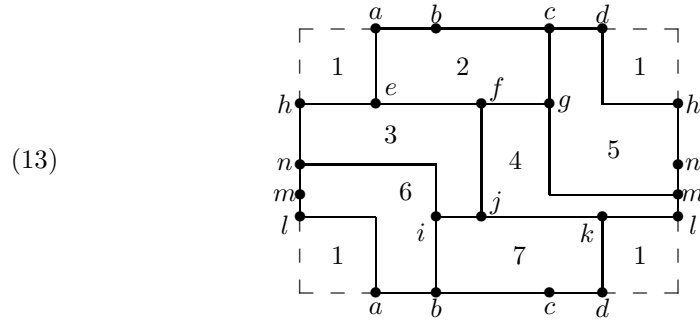
None of the sign assignments is all positive (or all negative).

Note that the fact that there are only four vertices in the color graph of 21211 can be argued by zero sets. The zero set is the union of two pentagons. In (6) they are the disjoint pentagons on the lower right and upper left. The four vertices not in the zero set are the vertices shown above.

14. HIGHER GENERA AND THOMPSON'S GROUP V

This section relates to the discussion in the last four paragraphs of Section 2 of [8].

14.1. Tree pairs on the torus. The following is Figure 2-16 on Page 35 of [19] with the addition of labels on the vertices.



The figure shows a map on a torus. The figure is to be interpreted as having the top edge of the rectangle identified with the bottom edge and the left edge of the rectangle identified with the right edge. Some parts of the top, bottom and sides are drawn as dashed lines since they are not part of the graph of the map and are interior to face 1. The point of the figure is that the graph cuts the torus into seven faces and each pair of faces shares an edge. Thus there is no coloring of the faces with less than seven colors.

(14)

It is simple to check that the color vector $\mathbf{c} = (1, 3, 1, 2, 2, 3, 1, 3)$ is valid for the left tree. If the colors are assigned on the right tree by using the numbering of the leaves as shown (and not the left-right ordering) so that leaf i gets color \mathbf{c}_i , then it is seen that the coloring is valid for the right tree as well. It follows that the graph in (13) has a proper, edge 3-coloring in spite of the fact that as a map it does not have a proper, face 4-coloring.

The smallest examples (by computer search) have 6 leaves for each tree, and our example is of this size. Examples are not rare. Of 13,800 triples involving trees with 6 leaves, there are 3,584 triples with no valid edge 3-coloring.

Below are two trees where the leaves of each tree are numbered to indicate the bijection between the two sets of leaves.

$$(15) \quad \left(\begin{array}{c} 0 \\ \swarrow \quad \downarrow \quad \searrow \\ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \end{array} , \begin{array}{c} 0 \\ \swarrow \quad \downarrow \quad \searrow \\ 1 \quad 4 \quad 3 \quad 6 \quad 2 \quad 5 \end{array} \right)$$

To discuss the possible colorings of the two trees we redraw the trees by labeling the leaves and vertices with letters. We use letters a through d to represent colors not yet assigned. We use x , y and z for assigned colors where the colors x , y and z represent different colors. We can assign the two colors on leaves 1 and 2 arbitrarily (as long as they are different) and we assign them colors x and y , respectively. We get the following picture.

$$\left(\begin{array}{c} 0 \\ \swarrow \quad \downarrow \quad \searrow \\ x \quad y \quad a \quad b \quad c \quad d \end{array} , \begin{array}{c} 0 \\ \swarrow \quad \downarrow \quad \searrow \\ x \quad b \quad a \quad d \quad y \quad c \end{array} \right)$$

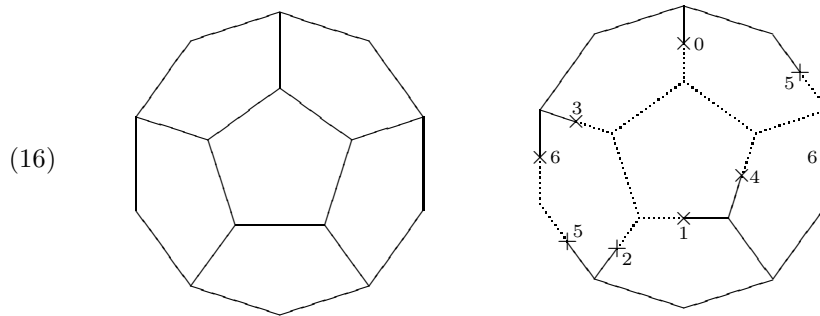
We have $c \neq d$, and the color b cannot equal $c + d$, so either $b = c$ or $b = d$. We will consider the two cases separately, but first we note facts that apply to either case. We have $a \neq x + y = z$, $c \neq d$, $b \neq x$, $a \neq d$, and $c \neq y$.

Case 1: $b = c$. Now $c \neq y$ and $c = b \neq x$ has $b = c = z$. With $b = c$, we get $b + c + d = d$ which cannot equal $x + y + a = z + a$. But $a \neq d$ so $d = z$. But this makes $c = d$ which cannot happen.

Case 2: $b = d$. Now $x + b \neq a + d = a + b$ so $a \neq x$. We also have $a \neq x + y = z$, so $a = y$. But now $b = d \neq a = y$ and $d = b \neq x$ so $b = d = z$. Now $b + c + d = c$ which cannot equal $x + y + a = z + a = z + y = x$. But $c \neq y$ so $c = z$. But now $c = d = z$ which cannot happen.

Thus (15) gives an example of an element of V with no valid coloring. We now show that this is a well known fact.

14.3. Tree pairs on the projective plane. Below on the left is the standard cubic “map” on RP^2 which has six pentagonal faces each of which shares an edge with all the others. Each point on the outer decagon is to be identified with its antipode. The graph is the Petersen graph (Figure 4-2 on Page 102 of [19]).



On the right, the map has been cut in 7 places, numbered 0 through 6, to break the map into two trees that meet in their leaves. It is easy to check that the two

trees and the numbering of the leaves is an embedding of the pair of trees in (15) so that the left tree in (15) maps to the tree in (16) drawn in dotted lines and the right tree in (15) maps to the tree in (16) drawn in solid lines. From the facts shown about (15), the map on the left in (16) has no proper, edge 3-coloring. This fact is standard. See the discussion on Page 102 of [19].

The map in (16) is not “Whitney” in several senses. While the map does break into a pair of trees, the break is not induced by a Hamiltonian circuit in the dual graph. The dual graph K_6 has plenty of Hamiltonian circuits, but they are all of length 6 and cannot break the graph in (16) into a pair of trees since it would only make 6 cuts. This would result in a loop in one of the two pieces. The dual graph also does not satisfy the hypotheses of Whitney’s theorem. The graph K_6 has 20 triangles, but in the embedding into RP^2 , only 10 of them bound faces. Similar comments apply to the map (13) on the torus.

15. ENUMERATION

In this section we record various results and questions about counts. Trees are counted by the Catalan numbers and we will mention those first. While the Catalan numbers have exponential growth, there is no exact formula for them specifically in terms of powers. The other counts we come across have such formulas. While it is not necessary to do so, it is often pleasant to arrive at these other formulas recursively. Thus we will also discuss well known generalizations of the Fibonacci numbers that we will encounter.

15.1. Trees. As mentioned in Section 4, the number of trees with n internal vertices is

$$C(n) = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$$

the n -th Catalan number. We have

$$\begin{aligned} \frac{C(n+1)}{C(n)} &= \frac{(2n+2)!}{(n+1)!(n+2)!} \frac{n!(n+1)!}{(2n)!} \\ &= \frac{(2n+2)(2n+1)}{(n+2)(n+1)} \\ &= 4 - \frac{6}{n+2}. \end{aligned}$$

Thus the Catalan numbers grow about as fast as 4^n after a somewhat slower start. Thus we can take it that the Catalan numbers will overtake any function growing no faster than k^n for $k < 4$.

15.2. Colorings of a single tree. If T is a tree with n internal vertices, then it has 2^n sign assignments. We count colorings modulo the action of S_3 on the colors, so we assume that the root color is 1 and that the top internal vertex is positive. Thus we get that the number of colorings of T modulo the action of S_3 on the colors is 2^{n-1} . This is equivalent to the formula for δ_n at the top of Page 212 in [26] where the action of S_3 is not taken into account.

Note that 2^{n-1} is smaller than the total number vectors of length $n+1$ with values in $\{1, 2, 3\}$ ignoring validity. This latter number is 3^{n+1} if the action of S_3 is ignored, or approximately $\frac{1}{2}(3^n)$ modulo the action of S_3 . We say “approximately” since not every vector uses all three colors.

Note also that these numbers are smaller than 4^n .

15.3. Recursive sequences. Given a quintuple of numbers (p, q, k, a, b) we can define a sequence recursively by $t(0) = a$, $t(1) = b$, and $t(n+1) = pt(n) + qt(n-1) + k$ for $n \geq 2$. When the quintuple is $(1, 1, 0, 0, 1)$, we have the Fibonacci sequence. For any quintuple with $k = 0$, it can be shown that the sequence is a linear combination of two special sequences, known as Lucas sequences, with quintuples $(p, q, 0, 0, 1)$ and $(p, q, 0, 2, p)$. The numbers $(1, 2, 0, 0, 1)$ define the Jacobsthal numbers $J(n)$ which we will encounter below. The Jacobsthal numbers are sequence A001045 of the Online Encyclopedia of Integer Sequences (OEIS). We will also encounter the sequence given by $(2, 3, 0, 0, 1)$ which is sequence A015518 of the OEIS.

We include the possibility of non-zero values for k because of the following which is easily proven by induction.

Lemma 15.1. *Let $t(n)$ be determined by $(p, q, 0, 0, b)$, and let*

$$s(n) = \sum_{i=0}^n t(i)$$

be the sequence of partial sums of the $t(i)$. Then the sequence $s(n)$ is determined by the quintuple $(p, q, b, 0, b)$.

In particular the partial sums $g(n)$ of the Fibonacci numbers satisfy $g(0) = 0$, $g(1) = 1$, and $g(n+1) = g(n) + g(n-1) + 1$ for $n \geq 2$. Writing out the first few of these numbers shows that they are closely related to the Fibonacci numbers themselves. This is not surprising since this fact is true of exponential sequences and the Fibonacci numbers are linear combination of two exponential sequences.

In general the behavior of the sequence corresponding to $(p, q, 0, a, b)$ is determined by the eigenvalues λ_1 and λ_2 of the matrix $\begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$. The terms of the sequence are then linear combinations of like powers of λ_1 and λ_2 .

The eigenvalues are roots of $\lambda(\lambda - p) - q = \lambda^2 - p\lambda - q$. So if q is already in the form $q = m(m - p)$, then the eigenvalues are $\lambda_1 = m$ and $\lambda_2 = p - m$. Thus we get particularly nice behavior from the sequences if $m = \pm 1$. For $m = -1$, we get $q = p + 1$ and for $m = 1$, we get $q = 1 - p$.

In the cases we said we would encounter, we have $q = p + 1$ and the following is easy to prove by induction.

Lemma 15.2. *Let $t(n)$ be determined by $(p, p+1, 0, 0, 1)$ and let $s(n)$ be the sequence of partial sums of the $t(i)$. Then*

$$(p+2)t(n) = (p+1)^n - (-1)^n,$$

$$ps(n) = t(n+1) - \frac{1}{2}(1 + (-1)^n).$$

for all $n \geq 0$.

Thus we see that the values of $t(n)$ and the partial sums of the $t(i)$ are dominated by powers of $p+1$.

For other values of p and q , the linear combinations of powers of the eigenvalues are more complicated.

It is because of Lemma 15.2 that we say that the recursive formulas are not necessary. However, the recursive formulas are a pleasant way to discover some of the enumeration formulas.

15.4. Acceptable colorings. Proposition 12.6 characterizes acceptable colorings of length at least two as those that are not constant and do not sum to zero. Every non-constant color vector of length $n + 1$ is of the form $1^i 2[123]^{n-i}$ if we mod out by the action of S^3 on the colors. We compute $c(n)$, the number of these that do not sum to zero.

We have $c(0) = 0$ and $c(1) = 1$.

We calculate $c(n + 1)$ by breaking the vectors of length $n + 2$ into three classes.

(I) There is one vector of the form $1^{i+1} 2$.

(II) There are $3c(n)$ vectors of the form $(1^i 2[123]^{n-i})x$, $x \in \{1, 2, 3\}$, where the sum of $(1^i 2[123]^{n-i})$ is not zero. But only two values of x give the whole vector a non-zero sum, so this contributes $2c(n)$ to $c(n + 1)$.

(III) This leaves vectors of the form $(1^i 2[123]^{n-i})x$ where the sum of $(1^i 2[123]^{n-i})$ is zero. But $(1^i 2[123]^{n-i})$ has the form $\mathbf{p}y$ where \mathbf{p} must have sum y . Now $(1^i 2[123]^{n-i})$ cannot have the form $1^n 2$ since $1^n 2$ does not have sum zero. Thus \mathbf{p} has the form $\mathbf{p} = (1^i 2[123]^{n-i-1})$ and there are $c(n - 1)$ of these. Since we assume $\mathbf{p}y$ has sum zero, there is only one choice for y given \mathbf{p} . Since the sum of $\mathbf{p}y$ is zero, all three values of x will give $\mathbf{p}yx$ a non-zero sum. So this contributes $3c(n - 1)$ to $c(n + 1)$.

We have shown the following.

Proposition 15.3. *Modulo the action of S_3 on the colors, there are $c(n)$ acceptable color vectors for the set of trees with n carets, where $c(0) = 0$, $c(1) = 1$, and $c(n + 1) = 2c(n) + 3c(n - 1) + 1$.*

From Lemma 15.1, this is the sequence of partial sums of the sequence determined by $(2, 3, 0, 0, 1)$.

15.5. Rigid colorings. To count rigid colorings, we count the color vectors that are described in Proposition 9.4. These are (modulo permutations of the colors) the non-constant vectors that sum to 1 with no prefix that sums to 3.

Proposition 15.4. *Let $r(n)$ be the number (modulo the action of S_3 on the colors) of positive rigid color vectors valid for trees with n carets. Then $r(1) = 1$, $r(2) = 2$ and $r(n + 1) = r(n) + 2r(n - 1) + 1$.*

Proof. The cases $n = 1$ and $n = 2$ are done by direct checking.

Consider the set S of vectors \mathbf{c} that are color vectors that each positively and rigidly color some tree with $n + 1$ carets with $n \geq 2$.

We break our set into three disjoint sets. The first consists of all vectors in S of the form $2s$, the second consists of all vectors in S of the form $13s$ and the third consists of all vectors in S of the form $11s$. In these, s is the remainder of the color vector. In the first two cases, the sum of s is 3, and in the third case the sum of s is 1.

We first assume that s is a constant sequence. If we have the form $11s$, the sum of the full vector is 0 if s has even length, so it must have odd length. But then the sum is 1 and s is the constant vector 1 making the full vector constant. So the form $11s$ cannot have s constant.

In the other two cases, the sum of s is 3 and must be the constant sequence 3 of odd length. We get one valid, positive, rigid vector with s constant from the form $2s$ if the length of the full vector is even, and we get one valid, positive rigid vector with s constant from the form $13s$ if the length of the full vector is odd.

This contributes 1 to the number of valid vectors no matter what the parity of the length of the full vector is.

From now on we assume that s is not constant.

In form $2s$, we know that s is non-constant, sums to 3 and cannot have a prefix summing to 1. This is the criterion for Proposition 9.4 if we interchange the roles of 1 and 3. Thus the number of such s is equal to $r(n)$.

In form $13s$, we know that s is non-constant, sums to 3 and cannot have a prefix summing to 1. Once again, we refer to Proposition 9.4 with the roles of 1 and 3 interchanged and we have that the number of such s is $r(n-1)$.

In form $11s$, we know that s is non-constant, sums to 1 and cannot have a prefix summing to 3. From Proposition 9.4, the number of such s is $r(n-1)$.

Thus we have $r(n+1) = r(n) + 2r(n-1) + 1$. \square

From Lemma 15.1, this is the sequence of partial sums of the Jacobsthal numbers. Note that in this case, the value of p in Lemma 15.2 is 1. So the values of the $r(n)$ are never far from the values of the Jacobsthal numbers.

15.6. Flexible colorings. Modulo the action of S_3 on the colors, the number of acceptable flexible colorings for trees of n internal vertices $f(n)$ satisfies $c(n) = f(n) + r(n)$. We have the following.

Lemma 15.5. *The count $f(n)$ satisfies $f(1) = 0$, $f(2) = 1$ and $f(n+1) = 3f(n) + r(n)$.*

Proof. The known values of $c(n)$ and $r(n)$ verify the base cases and the formula for $f(n+1)$ for $n = 1$ and $n = 2$.

From Propositions 15.3 and 15.4, we have

$$\begin{aligned} f(n+1) &= c(n+1) - r(n+1) \\ &= 2c(n) + 3c(n-1) - (r(n) + 2r(n-1)) \\ &= 2(c(n) - r(n)) + r(n) + 3(c(n-1) - r(n-1)) + r(n-1) \\ &= 2f(n) + r(n) + 3f(n-1) + r(n-1) \\ &= 2f(n) + r(n) + f(n) \\ &= 3f(n) + r(n). \end{aligned}$$

\square

15.7. Highly colorable maps. These last sections have many questions and few results.

We found much about map colorings by writing computer programs to look for colorings. It was natural to count the colorings as they were found. We discovered that certain maps had many more colorings than all other maps.

Let n be at least 2. We define $m_1(n)$, $m_2(n)$, $m_3(n)$ and $m_4(n)$ as those positive integers that are the largest possible with $m_4(n) < m_3(n) < m_2(n) < m_1(n)$ so that for each $i \in \{1, 2, 3, 4\}$ there is a tree pair (D_i, R_i) with n internal vertices each, corresponding to a map in \mathfrak{M} , and having $m_i(n)$ different valid color vectors (modulo the action of S_3 on the colors). Stated differently, $m_1(n)$ is the largest number of colorings of maps in \mathfrak{M} of that size ($n+2$ faces), $m_2(n)$ is the second largest, and so forth.

We can ask what the $m_i(n)$ are. We are unable to say what they are, but we can say what we suspect they are. We can also ask what maps achieve the numbers

that we think the $m_i(n)$ are. We are also unable to say what these maps are, but we can give some candidates. (The value $m_1(n)$ and corresponding map are now known. See the note after Question 15.8.)

15.7.1. *The estimates.* We give our observations. Counts depend heavily on the parity of n . For this reason we use the notation $f(n) + (e, o)$ to denote $f(n) + e$ when n is even and $f(n) + o$ when n is odd. The Jacobsthal numbers $J(n)$ are as introduced in Section 15.3. From Lemma 15.2, we have

$$J(n) = \frac{1}{3}(2^n + (-1)^{n+1}).$$

Calculations support the following.

Conjecture 15.6 (Color count). *The following hold.*

$$(17) \quad m_1(n) = J(n-3) + (1, 0), \quad n \geq 5,$$

$$(18) \quad m_2(n) = J(n-4) + (7, 5) = m_1(n-1) + (7, 4), \quad n \geq 7,$$

$$(19) \quad m_3(n) = J(n-4) + (1, 0) = m_1(n-1), \quad n \geq 7,$$

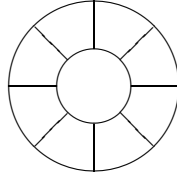
$$(20) \quad m_4(n) = J(n-4) - (1, 2) = m_1(n-1) - (1, 3), \quad n \geq 7.$$

Given the rapid growth of $J(n)$, it is seen that there is a huge gap in the number of colorings between the largest and second largest number of colorings.

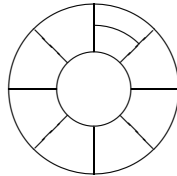
15.7.2. *The maps.* It is easier to discuss and calculate numbers of vertex colorings of maps than face colorings because of the availability of the chromatic polynomial. The duals to the maps in \mathfrak{W} are triangulations of the 2-sphere. We are thus interested in ways to describe and refer to triangulations of the 2-sphere. We describe some terminology and notation.

We let C_n be the polygon of n vertices and edges. If Γ and G are graphs, then ΓG is the join of Γ and G . This is the graph whose vertex set is $V(\Gamma) \cup V(G)$ and whose edge set is $E(\Gamma) \cup E(G) \cup T$ where T has an edge joining v to w for every (v, w) in $V(\Gamma) \times V(G)$. Then $C(\Gamma)$ is the “cone on Γ ” and is the join of Γ and the one point graph with no edges. We then get $\Sigma(\Gamma)$, the “suspension of Γ ,” which is the join of Γ and the two point graph with no edges.

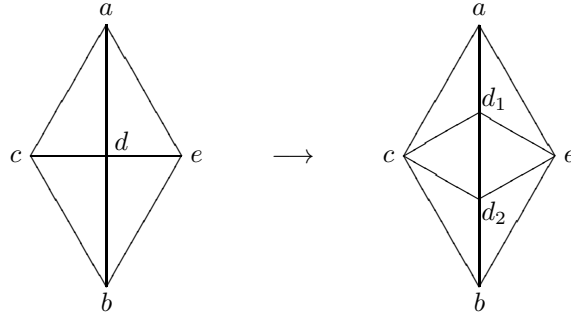
The graph we are most interested in is $W_{n+2} = \Sigma(C_n)$ or the biwheel of $n+2$ vertices. This graph can be embedded in the 2-sphere to give a triangulation of the 2-sphere. The dual of W_{10} is shown below.



We next describe four variations of W_n . The first is the one-bar variation Θ_{n+1} of W_n . Its name derives from the appearance of its dual. The dual of Θ_{11} is below.

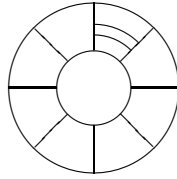


The difference between the graph Θ_{n+1} and W_n is explained by the transition pictured below.

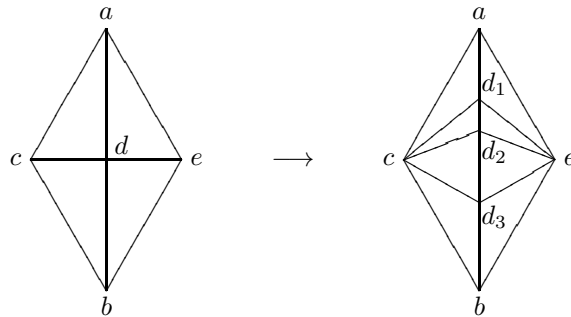


The left figure shows four triangles in W_n . The points a and b are the “suspension points” and the points c , d and e are three points in the polygon C_{n-2} . The graph Θ_{n+1} is created by replacing the left figure in W_n by the right figure and leaving the rest of W_n the same. Thus the transition raises the number of vertices by one and the number of triangles by two.

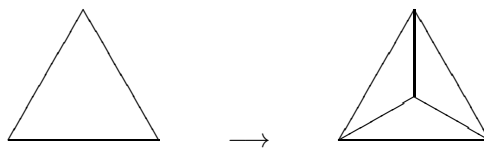
The second is the two-bar variation Ξ_{n+2} of W_n . The dual of Ξ_{12} is shown below.



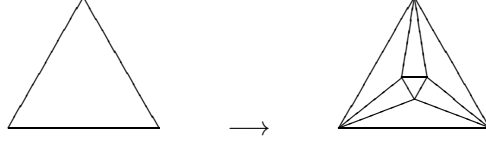
The graph Ξ_{n+2} is obtained by a replacement similar to that used to obtain Θ_{n+1} . The figure below left in W_n is replaced by the figure below right.



The next two variations on W_n are best described without reference to the original dual map. Each is obtained by adding new edges to a single triangular face in W_n . The graph Y_{n+1} is obtained by the following modification.



The graph ∇_{n+3} is obtained by the following modification.



15.7.3. *The counts.* We are interested in the number of proper, vertex 4-colorings of W_n , Θ_n , Ξ_n , Y_n and ∇_n . Available to us is the chromatic polynomial $P(\Gamma, t)$ of a finite graph Γ and the standard techniques for calculating the polynomial. We are thus interested in $P(\Gamma, 4)$ for Γ one of the five graphs listed. Note that we have been counting colorings modulo the permutations of the colors while $P(\Gamma, t)$ considers colorings different even if one is obtained from the other by permuting the colors. This is not a problem since none of our graphs has a proper, 2-coloring. It follows from this that the number we want is $1/24$ of the number that the chromatic polynomial gives.

We state certain results without giving details. The calculations using chromatic polynomials are straightforward. They are a bit longer in the case of ∇_n and Ξ_n , but there are no surprises. We have

$$(21) \quad \frac{1}{24}P(W_n, 4) = \frac{1}{3}(2^{n-3} + (-1)^n) + \frac{1}{2}(1 + (-1)^n),$$

$$(22) \quad \frac{1}{24}P(\Theta_n, 4) = \frac{1}{3}(2^{n-5} + (-1)^n) + \frac{1}{2}(4 + 4(-1)^{n-1}),$$

$$(23) \quad \frac{1}{24}P(\Xi_n, 4) = \frac{1}{3}(2^{n-4} + (-1)^{n-1}) + \frac{1}{2}(5 + 9(-1)^n),$$

$$(24) \quad \frac{1}{24}P(Y_n, 4) = \frac{1}{24}P(W_{n-1}, 4),$$

$$(25) \quad \frac{1}{24}P(\nabla_n, 4) = \frac{1}{3}(2^{n-3} + (-1)^{n-1}) + \frac{1}{2}(4 + 6(-1)^{n-1}).$$

The fourth equality is a triviality given the nature of the relation between Y_n and W_{n-1} .

It is a bit more revealing to compare each quantity on the left with a corresponding quantity of some W_j as is done with Y_n . We do this below where, as in Paragraph 15.7.1, we replace expressions such as $\frac{1}{2}(a + b(-1)^n)$ with a pair of numbers (e, o) where e is the value of the expression when n is even and o is the value when n is odd. Certain symmetries wreak havoc with comparisons for low values of n , and so restricting to $n \geq 7$ gives

$$\frac{1}{24}P(\Theta_n, 4) = \frac{1}{24}P(W_{n-2}, 4) + (-1, 4),$$

$$\frac{1}{24}P(\Xi_n, 4) = \frac{1}{24}P(W_{n-1}, 4) + (7, -3),$$

$$\frac{1}{24}P(Y_n, 4) = \frac{1}{24}P(W_{n-1}, 4) + (0, 0),$$

$$\frac{1}{24}P(\nabla_n, 4) = \frac{1}{24}P(W_{n-1}, 4) + (-1, 4).$$

Given the exponential growth of $P(W_n, 4)$, it is seen that the number of colorings of the Θ_n are much smaller than the others. Comparing the above with (17)–(20), we see that Ξ_n gives what we guess to be the second largest value when n is even

and the fourth largest value when n is odd. The same role with the parity of n reversed is played by ∇_n . The graph Y_n always gives our guess at the third largest value.

The exponential growth of $P(W_n, 4)$ also shows that if our guesses at the $m_i(n)$ are correct, then there is a huge gap between largest number $m_1(n)$ of colorings of an n -vertex triangulation of the 2-sphere and the next number of colorings encountered which cluster around approximately $\frac{1}{2}m_1(n)$. Computer calculations hint that there might be other gaps among the number of colorings that grow with n , but they are no where near as large. For $n = 12$, the difference of $m_1(12)$ and $m_2(12)$ is 80 while the next largest gap is 12.

15.7.4. Questions. The formulas for the $m_i(n)$ in (17)–(20) are guesses and have been verified by us for values of n through $n = 12$ and by others through $n = 15$. The counts of the number of colorings of the graphs W_n , Θ_n , Ξ_n , Y_n and ∇_n in (21)–(25) have been formally derived.

Question 15.7. *Are the values for the $m_i(n)$, $1 \leq i \leq 4$, given in (17)–(20) correct?*

Computational evidence suggests that the biwheel has the most proper, vertex 4-colorings of any n -vertex triangulation of the 2-sphere. Through $n = 12$, no other triangulations have as many colorings.

Question 15.8. *Is W_n the only triangulation of the 2-sphere with $J(n-3) + \frac{1}{2}(1+(-1)^n)$ colorings?*

Note: Question 15.7 for $i = 1$ and Question 15.8 have been answered in the affirmative by Paul Seymour [20].

We have no evidence one way or another that the triangulations Ξ_n , Y_n and ∇_n are the only triangulations having the numbers of proper, vertex 4-colorings that they do. But we can still ask the question.

Question 15.9. *Are there n -vertex triangulations of the 2-sphere not isomorphic to Ξ_n , Y_n or ∇_n with $m_i(n)$ colorings for $2 \leq i \leq 4$?*

15.8. Zero sets and color graphs. This section also raises more questions than it answers.

If \mathbf{c} is a color vector of length $n+1$, then it has a zero set and complementary color graph on A_d with $d = n-1$. Thus small zero sets should go with large color graphs and vice versa. We can measure zero sets directly by the number of vertices of A_d that are in it and indirectly by the number of codimension-1 faces that are in it. We have few examples from calculations, but we have observed that diameter as well as size enters into this discussion. The color graphs of highest diameter we have seen are the color graphs of Section 10.3. It turns out that the complementary zero sets of these color graphs are the smallest we have seen measured in terms of codimension-1 faces. On the other hand, color graphs of smallest diameter have complementary zero sets that have the largest number of codimension-1 faces. We now describe some specifics.

We get two numbers to consider from Section 10. Let $n = d+1$. If \mathbf{c} is a color vector of length $n+1$, then it has a zero set which is determined by certain set $Z_{\mathbf{c}}$ of intervals in $[0, n]$. An interval J is in $Z_{\mathbf{c}}$ if the sum of the \mathbf{c}_i for $i \in J$ is zero. Each such $J \in Z_{\mathbf{c}}$ determines a codimension-1 face of A_d all of whose vertices are

trees for which \mathbf{c} is not valid. We define $u(n)$ to be the smallest integer and $l(n)$ to be the largest integer so that for any acceptable color vector \mathbf{c} of length $n + 1$, we have $l(n) \leq |Z_{\mathbf{c}}| \leq u(n)$.

We claim that $u(n) = \lfloor \frac{n^2}{4} \rfloor$, and we think that $l(n) = \lfloor \frac{n^2}{8} \rfloor$. We indicate the argument for $u(n)$ and give support for $l(n)$.

Lemma 15.10. *If \mathbf{k} is a constant vector of length n , then $|Z_{\mathbf{k}}| = \lfloor \frac{n^2}{4} \rfloor$.*

This follows from direct calculation and the fact that the intervals in $Z_{\mathbf{k}}$ are just the intervals of even length. Of course \mathbf{k} is not acceptable, but the fact is useful.

Proposition 15.11. *With definitions as above, we have $u(n) = \lfloor \frac{n^2}{4} \rfloor$.*

Proof. We only sketch the proof. We take an acceptable vector \mathbf{c} of length $n + 1$ and let \mathbf{p} and \mathbf{s} be prefix and suffix of \mathbf{c} such that $\mathbf{c} = \mathbf{ps}$ and so that \mathbf{p} is the longest constant prefix. We can assume that $\mathbf{p} = 1^j$ and that \mathbf{s} starts with 2 and has length $k + 1$ with $j + k = n$.

Since \mathbf{p} and \mathbf{s} are adjacent, we are interested in terminal intervals for \mathbf{p} , those intervals in \mathbf{p} that include the last position in \mathbf{p} , and initial intervals for \mathbf{s} , those intervals in \mathbf{s} that include the first position of \mathbf{s} .

By Lemma 15.10 and an inductive hypothesis, we have

$$|Z_{\mathbf{c}}| \leq \left\lfloor \frac{j^2}{4} \right\rfloor + \left\lfloor \frac{k^2}{4} \right\rfloor + AB + CD$$

where A is the number of terminal intervals in \mathbf{p} summing to zero, B is the number of initial intervals in \mathbf{s} summing to zero, C is the number of terminal intervals in \mathbf{p} summing to one, and D is the number of initial intervals \mathbf{s} summing to one. By taking into account the parities of j and k , and the acceptability of \mathbf{c} , one shows on a case by case basis that $AB + CD$ is no more than $\frac{2jk}{4}$. \square

That $l(n) = \lfloor \frac{n^2}{8} \rfloor$ is supported by computer calculation for small values of n and by the following fact. If we use a , b and c to represent the colors 1, 2, 3 in some ordering, then in any interval of length 4 one of the patterns aa , abc or $abab$ will occur. Thus in any interval of length 4 there is a subinterval of length at least two that sums to zero.

Lemma 15.10 immediately gives us examples of extremes. Let $\mathbf{c} = 1^n 2$, $\mathbf{d} = 1^k 2 1^k$ and $\mathbf{e} = 1^k 2 1^{k+1}$. Let $n = 2k$ for \mathbf{d} and let $n = 2k + 1$ for \mathbf{e} . It follows directly from Lemma 15.10 that the following are true.

$$|Z_{\mathbf{c}}| = \left\lfloor \frac{n^2}{4} \right\rfloor, \quad |Z_{\mathbf{d}}| = \left\lfloor \frac{n^2}{8} \right\rfloor, \quad |Z_{\mathbf{e}}| = \left\lfloor \frac{n^2}{8} \right\rfloor.$$

The relevance of these examples to this section is that \mathbf{c} is valid for only one tree (the right vine), and \mathbf{d} and \mathbf{e} are examples from Section 10.3 of color vectors whose color graphs have very high diameter.

Other vectors that are valid for only one tree are of the form $1^j 2 3 1^{n-j-1}$ whose trees are root shifts of the right vine, and 12^n whose tree is the left vine and also a root shift of the right vine. Thus all these have zero sets with $u(n)$ codimension-1 faces.

Another color vector that seems to realize $l(n)$ is $\mathbf{f} = (12)^k$. With $n = 2k - 1$, preliminary calculations hint that $|Z_{\mathbf{f}}| = \lfloor \frac{n^2}{8} \rfloor$. We have not investigated the color graph of \mathbf{f} .

We are left with the following questions.

Question 15.12. *With $n = d + 1$, is $\lfloor \frac{n^2}{2} \rfloor$ the highest possible diameter of a color set in A_d ?*

Question 15.13. *Is $l(n) = \lfloor \frac{n^2}{8} \rfloor$?*

Question 15.14. *Are there other color vectors realizing either $u(n)$ or $l(n)$?*

We end with even vaguer questions.

Question 15.15. *What is the nature of the color graph of $(12)^k$?*

Question 15.16. *What is the relationship between the shape of the color graph and the number and arrangement of the codimension-1 faces of the zero set of a color vector?*

For rigid coloring vectors (such as $1^n 2$), we can ask the following.

Question 15.17. *What is the relationship between the number of vertices in the (totally disconnected) color graph of a rigid color vector and the number of codimension-1 faces of its zero set?*

We end this section with one minor comment. The only vertices completely interior to the color graph of a color vector \mathbf{c} are those trees that are either completely positively signed by \mathbf{c} or are completely negatively signed by \mathbf{c} . If we accept that such signings of a tree by a given color vector are rare, then we have that most vertices in a color graph are on its boundary.

16. THE END

This section represents the ill defined frontier of this paper. It gathers questions that fit nowhere else.

Question 16.1. *Given vertices u and v in a color graph Γ in A_d , is there a pair of codimension-1 faces in A_d that contains a path in Γ between u and v ? Can the path and faces be chosen so that there is an intermediate vertex w in the path with the path from u to w in one face and the path from w to v in the other?*

Recall from Section 10 that for a color vector \mathbf{c} with $n + 1$ entries, the set $Z_{\mathbf{c}}$ is the set of intervals $[m, n]$ in $[0, n]$ for which the sum of the \mathbf{c}_i with $i \in [m, n]$ is zero.

Question 16.2. *Is there a way to characterize those sets of intervals in $[0, n]$ that are of the form $Z_{\mathbf{c}}$ for some acceptable color vector \mathbf{c} with $n + 1$ entries?*

Question 16.3. *Is there any significance to the inconsistent elements of E ?*

It is not clear what is meant by this last question. Perhaps there are imaginary colorings of some tree pairs.

Another question is motivated by our construction of $\Sigma(w)$. Its relevance to the paper is highly questionable. Let Γ be a finite graph. If the edges of Γ are numbered e_1, e_2, \dots, e_n with $n = |E(\Gamma)|$, then we turn Γ into a signed graph much as we do for $\Sigma(w)$. We let Γ_i be the subgraph of Γ having exactly the edges $\{e_k \mid 1 \leq k \leq i\}$. We say that edge e_{i+1} with endpoints a_i and b_i is positive if the parities in Γ_i of a_i and b_i agree, and is negative otherwise. Obviously, the signing of the edges depends on the numbering of the edges. We are interested in whether the signed graph Γ is balanced. The question we raise is as follows.

Question 16.4. Which connected, finite graphs become balanced for every numbering of their edges, which connected, finite graphs become balanced for no numbering of their edges, and which connected, finite graphs are not in the two classes just described?

Of course trees, having no interesting closed walks, are in the first class. Eliminating trees, a minor amount of experimentation hints that in the first class are the boundaries of polygons with an even number of edges, in the second class are the boundaries of polygons with an odd number of edges, and in the third class is everything else.

REFERENCES

1. K. Appel and W. Haken, *Every planar map is four colorable. I. Discharging*, Illinois J. Math. **21** (1977), no. 3, 429–490. MR MR0543792 (58 #27598a)
2. K. Appel, W. Haken, and J. Koch, *Every planar map is four colorable. II. Reducibility*, Illinois J. Math. **21** (1977), no. 3, 491–567. MR MR0543793 (58 #27598b)
3. J. W. Cannon, W. J. Floyd, and W. R. Parry, *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2) **42** (1996), no. 3-4, 215–256. MR 98g:20058
4. Rui Pedro Carpentier, *On signed diagonal flip sequences*, European J. Combin. **32** (2011), no. 3, 472–477. MR 2764810 (2012a:52040)
5. Bobbe Cooper, Eric Rowland, and Doron Zeilberger, *Toward a language theoretic proof of the four color theorem*, Adv. in Appl. Math. **48** (2012), no. 2, 414–431. MR 2873886
6. Sebastian A. Csar, Rik Sengupta, and Warut Suksompong, *On a subposet of the tamari lattice*, ArXiv preprint: <http://front.math.ucdavis.edu/1108.5690>, 2011.
7. Patrick Dehornoy, *Geometric presentations for Thompson's groups*, 2005, pp. 1–44. MR MR2176650
8. Shalom Eliahou, *Signed diagonal flips and the four color theorem*, European J. Combin. **20** (1999), no. 7, 641–646. MR 1721923 (2001d:05058)
9. Shalom Eliahou and Cédric Lecouvey, *Signed permutations and the four color theorem*, Expo. Math. **27** (2009), no. 4, 313–340. MR 2567026 (2011c:05008)
10. Georges Gonthier, *Formal proof—the four-color theorem*, Notices Amer. Math. Soc. **55** (2008), no. 11, 1382–1393. MR 2463991 (2009j:05079)
11. Sylvain Gravier and Charles Payan, *Flips signés et triangulations d'un polygone*, European J. Combin. **23** (2002), no. 7, 817–821. MR 1932681 (2004a:05042)
12. Louis H. Kauffman, *Map coloring and the vector cross product*, J. Combin. Theory Ser. B **48** (1990), no. 2, 145–154. MR 91b:05078
13. S. I. Kryuchkov, *The four color theorem and trees*, preprint, I. V. Kurchatov Institute of Atomic Energy IAE-5537/1, 1992.
14. Jean-Louis Loday, *The YY game*, ArXiv preprint: <http://front.math.ucdavis.edu/1108.5639>, 2011.
15. Martin Markl, Steve Shnider, and Jim Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002. MR MR1898414 (2003f:18011)
16. Ralph McKenzie and Richard J. Thompson, *An elementary construction of unsolvable word problems in group theory*, Word problems: decision problems and the Burnside problem in group theory (Conf., Univ. California, Irvine, Calif. 1969; dedicated to Hanna Neumann) (F. B. Cannonito W. W. Boone and R. C. Lyndon, eds.), Studies in Logic and the Foundations of Math., Vol. 71, North-Holland, Amsterdam, 1973, pp. 457–478. MR MR0396769 (53 #629)
17. Lionel Pournin, *The diameters of associahedra*, ArXiv preprint: <http://front.math.ucdavis.edu/1207.6296>, 2012.
18. Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas, *The four-colour theorem*, J. Combin. Theory Ser. B **70** (1997), no. 1, 2–44. MR MR1441258 (98c:05065)
19. Thomas L. Saaty and Paul C. Kainen, *The four-color problem: Assaults and conquest*, second ed., Dover Publications Inc., New York, 1986. MR MR863420 (87k:05084)
20. Paul Seymour, *Proof of a conjecture of Bowlin and Brin on four-colouring triangulations*, preprint, 2012.

21. Steven Shnider and Shlomo Sternberg, *Quantum groups*, International Press, Cambridge, MA, 1993, From coalgebras to Drinfel'd algebras, A guided tour.
22. P. G. Tait, *Note on a theorem of the geometry of position.*, Trans. of Edinb. XXIX, (1880), 657–660 (English).
23. Robin Thomas, *An update on the four-color theorem*, Notices Amer. Math. Soc. **45** (1998), no. 7, 848–859. MR MR1633714 (99g:05082)
24. Dirk Vertigan, *The computational complexity of Tutte invariants for planar graphs*, SIAM J. Comput. **35** (2005), no. 3, 690–712 (electronic). MR 2201454 (2006k:68045)
25. Hassler Whitney, *A theorem on graphs*, Ann. of Math. (2) **32** (1931), no. 2, 378–390. MR MR1503003
26. ———, *A numerical equivalent of the four color map Problem*, Monatsh. Math. Phys. **45** (1936), no. 1, 207–213. MR 1550643

Department of Mathematics
State University of New York at Oneonta
Oneonta, NY
USA
email: gsbowlin@gmail.com

Department of Mathematical Sciences
State University of New York at Binghamton
Binghamton, NY 13902-6000
USA
email: matt@math.binghamton.edu

INDEX

- A_d , 16
- E , 33
- F , 27
- F^+ , 54
- F_4 , 39
- P^+ , 53
- P^r , 53
- P_u , 53
- $S \frown T$, 14
- T^σ , 22
- T_v , 14
- V , 56
- Z_c , 41
- $\Sigma(w)$, 44
- \overline{u} , 18
- \mathcal{T} , 13
- \mathfrak{W} , 3
- $[u]$, 18, 25
- $c(n)$, 60
- $f(n)$, 61
- $m_i(n)$, 61
- $r(n)$, 60
- 4-tree, 50
- above
 - for subtrees, 46
- acceptable
 - color vector, 21
 - characterization, 52
- address
 - of vertex in tree, 14
- adjacent
 - subtrees, 46
- ancestor
 - in tree, 12
- associahedron, 16
 - colored, 27
 - diameter, 7
 - dihedral group action, 19
 - dimension, 16
 - edge, 17
 - face, 17
- balanced
 - signed graph, 44
- binary
 - tree, 12
 - as union of caret, 29
 - standard model, 13
- binary trees
 - union, intersection, difference, 30
- biwheel, 62
- caret, 28
 - center vertex, 29
 - exposed, 29
- carets
 - tree union of, 29
- Catalan number, 17
- Catalan numbers, 58
- center vertex
 - caret, 29
- chain
 - of increasing multiplications, 37
 - of minimally increasing multiplications, 37
- children
 - in tree, 12
- chromatic polynomial, 64
- code
 - for colors, 20
- color
 - code for, 20
 - element of group, 20
 - of leaf in tree, 20
 - of vertex in tree, 20
 - permutation, 23
 - effect on sign assignment, 23
 - root
 - of tree, 20
 - vector, 20
 - acceptable, 21
 - color graph of, 27
 - valid for pair, 21
 - valid for tree, 21
 - zero set of, 40
- color graph
 - of color vector, 27
- colored
 - associahedron, 27
 - vertices, 27
- coloring
 - compatible with sign structure, 45
 - normal, 23
 - of element of F , 34
 - of tree from vector, 20
- Compatibility Lemma, 35
- conjecture
 - Color count, 62
 - Injectivity, 49
 - signed path
 - first, 26
 - second, 45
 - Well definedness, 49
- deferment
 - of element of F , 54
- depth
 - vertex in tree, 38
- descendant
 - in tree, 12
- diameter
 - of associahedron, 7

- dihedral group
 - action on associahedron, 19
- dimension
 - associahedron, 16
 - face, 17
- drawing
 - of tree, 13
- dual
 - polygon to tree, 15
 - tree to polygon, 15
- dyadic
 - rational, 28
- edge
 - associahedron, 17
 - external in tree, 12
 - internal in tree, 12
 - leaf in tree, 12
 - root in tree, 12
 - rotation along, 18
- edge disjoint
 - subtrees, 15
- edge-face
 - coloring correspondence, 20
- exposed caret, 29
- face
 - of associahedron, 17
 - dimension, 17
 - structure, 17
- face-edge
 - coloring correspondence, 20
- Fibonacci numbers, 59
- flexible
 - sign assignment, 26
- graph
 - color
 - of color vector, 27
- group
 - of colors, 20
- group E , 33
- increasing
 - chain of multiplications, 37
 - multiplication, 37
- induction
 - Noetherian, 14
 - well founded, 14
- infix order
 - on tree, 13
- Jacobsthal numbers, 59
- leaves
 - left-right order, 13
 - tree, 12
 - color, 20
- left
 - subtree, 14
- vine, 35
- left-right order
 - on leaves, 13
- lemma
 - Compatibility, 35
- level
 - vertex in tree, 38
- local order
 - from planar embedding, 12
 - in tree, 12
- map
 - prime, 10
- minimally increasing
 - chain of multiplications, 37
 - multiplication, 37
- move
 - pentagonal, 47
 - square, 47
- multiplication
 - increasing, 37
 - minimally increasing, 37
- negative
 - normal coloring, 23
 - rigid
 - pattern, 53
- neighborhood
 - positive
 - of vertex, 54
- Noetherian
 - induction, 14
- normal
 - coloring, 23
 - negative, 23
 - positive, 23
- order
 - local in tree, 12
 - well founded, 14
- P-compatible
 - pair of pairs, 53
 - tree pair, 53
- pair
 - of finite trees, 11
 - P-compatible, 53
 - parity condition, 38
 - reduced, 29
 - reduction, 29
 - triangulated
 - polygon, 11
- parent
 - in tree, 12
- parity condition
 - on pair of trees, 38
- path
 - sign consistent, 45
 - sign structure of, 45

- square class, 49
- square equivalence, 49
- valid
 - of rotations, 25
- pattern, 53
 - all positive, 54
 - negative rigid, 53
 - positive rigid, 53
 - shadow
 - of tree, 40
- pentagonal
 - move, 47
 - relation, 33
- permuting
 - colors, 23
- Petersen graph, 57
- pivot vertex
 - of rotation, 18
- polygon
 - dual to tree, 15
 - triangulated
 - pair, 11
- positive
 - element of F , 37
 - monoid of F , 37
 - neighborhood
 - of vertex, 54
 - normal coloring, 23
 - pattern, 54
 - rigid
 - pattern, 53
- prefix order
 - on tree, 13
- prime
 - finite tree pair, 11
 - map, 10
- projection
 - of tree, 15
- projective plane
 - map needing six colors, 57
- rational
 - dyadic, 28
- reduced
 - tree pair, 29
- reduction
 - of tree pair, 29
- reflection
 - tree, 16
- regular expressions, 14
- relation
 - pentagonal, 33
 - square, 33
- right
 - subtree, 14
 - vine, 35
- rigid
 - pattern
 - negative, 53
 - positive, 53
 - sign assignment, 26
- root
 - color, 20
 - of tree, 12
- root shift
 - of tree, 16
- rooted
 - tree, 12
- rotation
 - action of, 31
 - on colored pairs, 36
 - along edge, 18
 - pivot vertex, 18
 - signed, 25
 - tree, 18
 - valid, 25
 - valid path, 25
- sequence
 - Catalan numbers, 58
 - Fibonacci numbers, 59
 - Jacobsthal numbers, 59
- shadow
 - interval
 - of vertex in tree, 40
 - of vertex in tree, 40
 - pattern of tree, 40
- sign
 - at vertex, 22
- sign assignment
 - compatible with sign structure, 45
 - effect of permutation, 23
 - flexible, 26
 - of tree, 22
 - rigid, 26
- sign consistent path, 45
- sign structure
 - compatible with coloring, 45
 - compatible with sign assignment, 45
 - of path, 45
 - vertex set, 46
- signed
 - rotation, 25
- square
 - equivalence of paths, 49
 - move, 47
 - path class, 49
 - relation, 33
- standard model
 - binary tree, 13
- subpattern, 53
- subtree, 13
 - above another subtree, 46
 - adjacent to a subtree, 46
 - left, 14
 - right, 14

- theorem
 - Trichotomy, 26
- Thompson's group F , 27
 - action on trees, 31
 - coloring of element, 34
 - deferment of element, 54
 - multiplication, 29
 - positive monoid, 37
 - presentations, 33
- Thompson's group V , 56
- torus
 - map needing seven colors, 55
- tree
 - ancestor, 12
 - binary, 12
 - as union of caret, 29
 - children, 12
 - coloring from vector, 20
 - definition, 12
 - descendant, 12
 - drawing, 13
 - dual to polygon, 15
 - external edge, 12
 - finite, 12
 - finite pair, 11
 - prime, 11
 - valid color vector, 21
 - infix order, 13
 - internal edge, 12
 - internal vertex, 12
 - leaf
 - color, 20
 - edge, 12
 - leaves, 12
 - locally finite, 12
 - locally ordered, 12
 - parent, 12
 - prefix order, 13
 - projection, 15
 - reflection, 16
 - root, 12
 - root edge, 12
 - root shift, 16
 - rooted, 12
 - rotation, 18
 - shadow
 - of vertex, 40
 - pattern, 40
 - shadow interval
 - of vertex, 40
 - sign assignment, 22
 - trivial, 12
 - valid color vector, 21
- vertex
 - color, 20
 - depth, 38
 - level, 38
 - shadow interval of, 40
- shadow of, 40
- sign, 22
- trees
 - binary
 - union, intersection, difference, 30
- triangulated
 - polygon
 - pair, 11
- Trichotomy Theorem, 26
- trivial
 - tree, 12
- valid
 - path of rotations, 25
 - rotation, 25
 - vector
 - for pair, 21
 - for tree, 21
- vector
 - color, 20
 - acceptable, 21
 - color graph of, 27
 - zero set of, 40
 - valid for pair, 21
 - valid for tree, 21
- vertex
 - center of caret, 29
 - of tree
 - depth, 38
 - internal, 12
 - level, 38
 - shadow, 40
 - shadow interval, 40
 - sign, 22
 - positive neighborhood, 54
 - tree
 - color, 20
- vertex set
 - of sign structure, 46
- vertices
 - colored, 27
- vine, 35
 - left, 35
 - right, 35
- well founded
 - induction, 14
 - order, 14
- Whitney
 - theorem, 3
- zero set
 - of color vector, 40